PROJECTORS AND HALL \(\pi\)-SUBGROUPS
IN FINITE \(\pi\)-SOLVABLE GROUPS

RODICA COVACI

Abstract. Let \(\pi\) be a set of primes and \(X\) be a \(\pi\)-closed Schunck class with the \(P\) property. The paper gives conditions with respect to which an \(X\)-projector \(H\) of a finite \(\pi\)-solvable group \(G\) is a Hall \(\pi\)-subgroup of \(G\), and consequently we have that \(N_G(N_G(H)) = N_G(H)\).

1. Preliminaries

All groups considered in the paper are finite. Let \(\pi\) be a set of primes, \(\pi'\) the complement to \(\pi\) in the set of all primes and \(O_{\pi'}(G)\) the largest normal \(\pi'\)-subgroup of a group \(G\).

We first give some useful definitions.

Definition 1.1. ([8], [11]) a) A class \(X\) of groups is a homomorph if \(X\) is epimorphically closed, i.e. if \(G \in X\) and \(N\) is a normal subgroup of \(G\), then \(G/N \in X\).

b) A group \(G\) is primitive if \(G\) has a stabilizer, i.e. a maximal subgroup \(H\) with \(\text{core}_GH = \{1\}\), where \(\text{core}_GH = \cap\{H^g/g \in G\}\).

c) A homomorph \(X\) is a Schunck class if \(X\) is primitively closed, i.e. if any group \(G\), all of whose primitive factor groups are in \(X\), is itself in \(X\).

Definition 1.2. a) A positive integer \(n\) is said to be a \(\pi\)-number if for any prime divisor \(p\) of \(n\) we have \(p \in \pi\).

b) A finite group \(G\) is a \(\pi\)-group if \(|G|\) is a \(\pi\)-number.
Definition 1.3. ([6]) A group $G$ is $\pi$-solvable if every chief factor of $G$ is either a solvable $\pi$-group or a $\pi'$-group. For $\pi$ the set of all primes, we obtain the notion of solvable group.

Definition 1.4. A class $X$ of groups is said to be $\pi$-closed if

$$G/O_{\pi'}(G) \in X \Rightarrow G \in X.$$ 

A $\pi$-closed homomorph, respectively a $\pi$-closed Schunck class is called $\pi$-homomorph, respectively $\pi$-Schunck class.

Definition 1.5. ([7], [8]) Let $X$ be a class of groups, $G$ a group and $H$ a subgroup of $G$.

a) $H$ is an $X$-maximal subgroup of $G$ if: (i) $H \in X$; (ii) $H \leq H^* \leq G$, $H^* \in X$ imply $H = H^*$.

b) $H$ is an $X$-projector of $G$ if, for any normal subgroup $N$ of $G$, $HN/N$ is $X$-maximal in $G/N$.

c) $H$ is an $X$-covering subgroup of $G$ if: (i) $H \in X$; (ii) $H \leq K \leq G$, $K_0 \leq K$, $K/K_0 \in X$ imply $K = HK_0$.

Definition 1.6. ([3], [4]) Let $X$ be a class of groups. We say that $X$ has the $P$ property if, for any $\pi$-solvable group $G$ and for any minimal normal subgroup $M$ of $G$ such that $M$ is a $\pi'$-group, we have $G/M \in X$.

The following results are used in this paper.

Theorem 1.7. ([1]) A solvable minimal normal subgroup of a group is abelian.

Theorem 1.8. ([1]) Suppose that $G$ has a $\neq \{1\}$ normal solvable subgroup and let $S$ be a maximal subgroup of $G$ with $\text{core}_G S = \{1\}$. Then, the existence of a $\neq \{1\}$ normal solvable subgroup of $S$ implies the existence of a normal subgroup $N \neq \{1\}$ of $S$ with $(|N|, |G : S|) = 1$.

Theorem 1.9. ([2]) a) Let $X$ be a class of groups, $G$ a group and $H$ a subgroup of $G$. If $H$ is an $X$-covering subgroup of $G$ or $H$ is an $X$-projector of $G$, then $H$ is $X$-maximal in $G$. 

18
b) If $X$ is a homomorph and $G$ is a group, then a subgroup $H$ of $G$ is an $X$-covering subgroup of $G$ if and only if $H$ is an $X$-projector in any subgroup $K$ of $G$ with $H \subseteq K$.

Theorem 1.10. Let $X$ be a homomorph.

a) ([7]) If $H$ is an $X$-covering subgroup of a group $G$ and $N$ is a normal subgroup of $G$, then $HN/N$ is an $X$-covering subgroup of $G/N$.

b) ([8]) If $H$ is an $X$-projector of a group $G$ and $N$ is a normal subgroup of $G$, then $HN/N$ is an $X$-projector of $G/N$.

c) ([7]) If $H$ is an $X$-covering subgroup of $G$ and $H \leq K \leq G$, then $H$ is an $X$-covering subgroup of $K$.

Theorem 1.11. ([5]) Let $X$ be a $\pi$-homomorph. The following conditions are equivalent:

(1) $X$ is a Schunck class;

(2) any $\pi$-solvable group has $X$-covering subgroups;

(3) any $\pi$-solvable group has $X$-projectors.

2. Hall $\pi$-subgroups in finite $\pi$-solvable groups

Of special interest in this paper will be the Hall $\pi$-subgroups and some of their properties. The Hall subgroups were given in [9]. Ph. Hall studied them in finite solvable groups. In [6], S. A. Ćunihić extended this study to finite $\pi$-solvable groups.

Definition 2.1. Let $G$ be a group and $H$ a subgroup of $G$.

a) $H$ is a $\pi$-subgroup of $G$ if $H$ is a $\pi$-group.

b) $H$ is an Hall $\pi$-subgroup of $G$ if: (i) $H$ is a $\pi$-subgroup of $G$;

(ii) $|H|, |G : H| = 1$, i.e. $|G : H|$ is a $\pi'$-number.

We shall use some properties of the Hall $\pi$-subgroups given in [10]:

Theorem 2.2. ([10]) (Ph. Hall, S. A. Ćunihić) If $G$ is a $\pi$-solvable group, then:

a) $G$ has Hall $\pi$-subgroups and $G$ has Hall $\pi'$-subgroups;
b) any two Hall $\pi$-subgroups of $G$ are conjugate in $G$; any two Hall $\pi'$-subgroups of $G$ are conjugate in $G$ too.

**Theorem 2.3.** ([10]) Let $G$ be a group and $H$ an Hall $\pi$-subgroup of $G$.

a) If $H \leq K \leq G$, then $H$ is an Hall $\pi$-subgroup of $K$.

b) If $N$ is a normal subgroup of $G$, then $HN/N$ is an Hall $\pi$-subgroup of $G/N$.

We complete these properties with two new ones, which will be used in the formation theory considerations in the main section of this paper.

**Theorem 2.4.** Let $G$ be a $\pi$-solvable group, $H$ a subgroup of $G$ and $N$ a normal subgroup of $G$. If $HN/N$ is an Hall $\pi$-subgroup of $G/N$ and $H$ is an Hall $\pi$-subgroup of $HN$, then $H$ is an Hall $\pi$-subgroup of $G$.

**Proof.**

(i) $H$ is a $\pi$-subgroup of $G$, since $H$ is a $\pi$-subgroup of $HN$.

(ii) We shall prove that $|G : H|$ is a $\pi'$-number. Indeed, we know that $|G : HN| = |G/N : HN/N|$ is a $\pi'$-number. Further, $|HN : H|$ is a $\pi'$-number too. Then $|G : H| = |G : HN||HN : H|$ is a $\pi'$-number. □

**Theorem 2.5.** If $G$ is a $\pi$-solvable group and $H$ is a Hall $\pi$-subgroup of $G$, then $N_G(N_G(H)) = N_G(H)$.

**Proof.** We know that

$$N_G(H) = \{g \in G/H^g = H\} \supseteq H$$

and so we have $N_G(H) \subseteq N_G(N_G(H))$. We now prove that $N_G(N_G(H)) \subseteq N_G(H)$. Let $x \in N_G(N_G(H))$. It is known that $N_G(H) \unlhd N_G(N_G(H))$. It follows that $N_G(H)^x = N_G(H)$, hence $H^x \subseteq N_G(H)^x = N_G(H)$, which implies by 2.3.a) that $H$ and $H^x$ are Hall $\pi$-subgroups of $N_G(H)$. Applying Hall-Čunihin Theorem 2.2.b), we obtain that $H$ and $H^x$ are conjugate in $N_G(H)$. So there is an element $y \in N_G(H)$ such that $(H^x)^y = H$. It follows that $H^{xy} = H$, hence $xy \in N_G(H)$. But $y \in N_G(H)$ implies $y^{-1} \in N_G(H)$ and so $x = (xy)y^{-1} \in N_G(H)$. □
3. Projectors which are Hall $\pi$-subgroups in finite $\pi$-solvable groups

In [8], W. Gaschütz gives for finite solvable groups the following result: If $X$ is a Schunck class, $G$ a solvable group and $S$ an $X$-projector of $G$ such that $S$ is a $p$-group, then $S$ is a Sylow $p$-subgroup of $G$.

It is the aim of this paper to study similar properties in the more general case of finite $\pi$-solvable groups.

All groups considered in this section are finite $\pi$-solvable.

**Theorem 3.1.** Let $X$ be a $\pi$-Schunck class with the P property. If $G$ is a $\pi$-solvable group, such that there is a minimal normal subgroup $M$ of $G$ which is a $\pi'$-group, and if $H$ is an $X$-projector of $G$ which is a $\pi$-group, then $H$ is an Hall $\pi$-subgroup of $G$.

**Proof.** We will show that $|G : H|$ is a $\pi'$-number. Let $M$ be a minimal normal subgroup of $G$, such that $M$ is a $\pi'$-group. We know that $X$ has the P property, and so, by 1.6., we have $G/M \in X$.

On the other side, $H$ being an $X$-projector of $G$, we have, by 1.10., that $HM/M$ is an $X$-projector of $G/M$. Now 1.9.a) implies that $HM/M$ is $X$-maximal in $G/M$. But $G/M \in X$. It follows that $HM/M = G/M$, hence $HM = G$. From this and from $HM/M \cong H/H \cap M$, we obtain that


Since $|M : H \cap M|$ divides $|M|$ which is a $\pi'$-number, we obtain that $|M : H \cap M|$ is also a $\pi'$-number. Hence $|G : H|$ is a $\pi'$-number. □

In order to renounce to the condition on the group $G$ of having a minimal normal subgroup $M$ which is a $\pi'$-group, the next theorem contains the assumption that $H$ is an $X$-covering subgroup of $G$. This means, by 1.9.b), that $H$ is a particular $X$-projector.

**Theorem 3.2.** Let $X$ be a $\pi$-Schunck class with the P property. If $G$ is a $\pi$-solvable group and $H$ is an $X$-covering subgroup of $G$ which is a $\pi$-group, then $H$ is an Hall $\pi$-subgroup of $G$. 

21
**Proof.** By induction on $|G|$. We consider two cases:

1) There is a minimal normal subgroup $M$ of $G$, such that $M$ is a $\pi'$-group. By 1.9.b), $H$ is an $\aleph$-projector of $G$. Applying theorem 3.1., it follows that $H$ is an Hall $\pi$-subgroup of $G$.

2) Any minimal normal subgroup $M$ of $G$ is a solvable $\pi$-group. Hence, by 1.7., $M$ is abelian. If $H = G$, it follows from $H$ $\pi$-group that $H$ is an Hall $\pi$-subgroup of $G = H$. Let now $H \neq G$. We distinguish two possibilities:

a) For any minimal normal subgroup $M$ of $G$ we have $HM = G$.

Let us first prove that $H$ is a maximal subgroup of $G$. Indeed, we have $H < G$. Further, if $H \leq H^* < G$, we prove that $H = H^*$. Suppose that $H < H^*$, and let $h^* \in H^* \setminus H$. Let $M$ be a minimal normal subgroup of $G$. By the above, we have that $M$ is abelian and $G = HM$. So $h^* = hm$, where $h \in H$, $m \in M$. It follows that $m = h^{-1}h^* \in M \cap H^*$. Let us prove that $M \cap H^* = \{1\}$. Suppose that $M \cap H^* \neq \{1\}$. We have $M \cap H^* \subseteq H^*$. Further, $M \cap H^* \subseteq G$, since if $x \in G = HM = H^*M = MH^*$ and $m \in M \cap H^*$, then $x = m_1h^*$, where $m_1 \in M$, $h^* \in H^*$, and $M$ being abelian, we have:

$$x^{-1}mx = (m_1h^*)^{-1}m(m_1h^*) = (h^*)^{-1}m_1^{-1}mm_1h^* = (h^*)^{-1}mm_1^{-1}m_1h^* =$$

$$= (h^*)^{-1}mh^* \in M \cap H^*.$$ 

So $M \cap H^* \subseteq G$, $M \cap H^* \subseteq M$, $M \cap H^* \neq \{1\}$. But $M$ is a minimal normal subgroup. Hence $M \cap H^* = M$, which implies that $M \subseteq H^*$ and so $G = H^*M = H^*$, a contradiction with $H^* < G$. It follows that $M \cap H^* = \{1\}$. Hence $m = 1$ and so $h^* = h \in H$, in contradiction with the choice of $h^*$. We proved that $H = H^*$. So $H$ is a maximal subgroup of $G$.

Let us notice that $core_GH = \{1\}$. Indeed, if we suppose that $core_GH \neq \{1\}$, it follows since $core_GH \trianglelefteq G$ that there exists a minimal normal subgroup $M$ of $G$ such that $M \subseteq core_GH$. We obtain $G = HM \subseteq Hcore_GH = H$, in contradiction with $H \neq G$. So $core_GH = \{1\}$.

We are now in the hypotheses of theorem 1.8. By 1.8., it follows the existence of a normal subgroup $N \neq \{1\}$ of $H$, such that $([N], |G : H|) = 1$. But $H$ being a
π-group, \( N \) is also a π-group. Then \(|G : H|\) is a π'-number. It follows that \( H \) is an Hall π-subgroup of \( G \).

b) There is a minimal normal subgroup \( M \) of \( G \) such that \( HM \neq G \).

We apply the induction to the π-solvable group \( HM \), with \(|HM| < |G|\). By \ref{1.10.c}, \( H \) is an \( \pi \)-covering subgroup of \( HM \). Further, \( H \) is a π-group. By the induction, \( H \) is an Hall π-subgroup of \( HM \).

We now apply the induction to the π-solvable group \( G/M \), with \(|G/M| < |G|\). By \ref{1.10.a}, \( HM/M \) is an \( \pi \)-covering subgroup of \( G/M \). Further, we have that \(|HM/M| = |H/H \cap M|\) divides \(|H|\), and so \( HM/M \) is a π-group. By the induction, \( HM/M \) is an Hall π-subgroup of \( G/M \).

Finally, theorem 2.4. leads us to the conclusion that \( H \) is an Hall π-subgroup of \( G \). □

**Corollary 3.3.** Let \( \pi \)-Schunck class with the P property. If \( G \) is a π-solvable group and \( H \) is an \( \pi \)-covering subgroup of \( G \) which is a π-group, then \( N_G(N_G(H)) = N_G(H) \).

**Proof.** Follows from 3.2. and 2.5.. □

**References**


Babeș-Bolyai University, Str. Kogălniceanu 1, Cluj-Napoca, Romania

E-mail address: rcovaci@math.ubbcluj.ro