SMOOTH DEPENDENCE OF SOLUTION ON PARAMETERS FOR THE VOLTERRA-FREDHOLM INTEGRAL EQUATION

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Abstract. In this paper we will give conditions that ensures the differentiability with respect to parameters of the solution of Volterra-Fredholm nonlinear integral equation.

1. Introduction

In the present paper consider the nonlinear integral equation of Volterra-Fredholm type:

$$u(x, t) = f(x, t) + \int_0^t \int_a^b K(x, t, y, s, u(y, s))dyds$$

(1)

∀t ∈ [0, c], ∀x ∈ [α, β], where [a, b] ⊂ [α, β]. Applying fiber Picard operators theory, we will prove the differentiability of the solution of (1) with respect to a and b.

2. Fiber Picard operators

Let (X, d) be a metric space and A : X → X an operator. In this paper we will use the following notations:

$$F_A := \{x \in X : A(x) = x\};$$

$$A^0 := 1_X, A^{n+1} := A \circ A^n \forall n \in \mathbb{N}.$$
Definition 2.1. (I. A. Rus [1]) The operator $A$ is said to be:

(i) weakly Picard operator (wPo) if $\forall x_0 \in X A^n(x_0) \to x_0^*$, and the limit $x_0^*$ is a fixed point of $A$, which may depend on $x_0$.

(ii) Picard operator (Po) if $F_A = \{x^*\}$ and $\forall x_0 \in X A^n(x_0) \to x^*$.

In the next section we need the following result:

Theorem 2.1. (Fiber Contraction Principle, I. A. Rus [3]) Let $(X, d)$, $(Y, \rho)$ be two metric spaces and $B : X \to X$, $C : X \times Y \to Y$ two operators such that:

(i) $(Y, \rho)$ is complete;

(ii) $B$ is a Picard operator, $F_B = \{x^*\}$;

(iii) $C(\cdot, y) : X \to Y$ is continuous $\forall y \in Y$;

(iv) $\exists a \in [0, 1]$ such that the operator $C(x, \cdot) : Y \to Y$ is an $a$-contraction for all $x \in X$; let $y^*$ be the unique fixed point of $C(x^*, \cdot)$.

Then

\[ A : X \times Y \to X \times Y, \quad A(x, y) := (B(x), C(x, y)) \]

is a Picard operator and $F_A = \{(x^*, y^*)\}$.

This theorem is very useful for proving solutions of operatorial equations to be differentiable with respect to parameters. For such results see [6], [3], [2], [4], [5], etc.

3. Main result

Theorem 3.1. Consider the equation (1) in the next conditions:

(i) $f \in C([a, b] \times [0, c])$ and $K \in C([a, b] \times [0, c] \times [a, b] \times [0, c] \times \mathbb{R})$;

(ii) there exists $L_K > 0$ such that:

\[ |K(x, t, y, s, u) - K(x, t, y, s, v)| \leq L_K |u - v| \]

$\forall (x, t, y, s) \in [a, \beta] \times [0, c] \times [a, \beta] \times [0, c]$, $\forall u, v \in \mathbb{R}$.

Then:

a) for all $a < b \in [\alpha, \beta]$, the equation (1) has in $C([\alpha, \beta] \times [0, c])$ a unique solution
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\( u^*(\cdot, \cdot, a, b). \)

b) for all \( u_0 \in C([\alpha, \beta] \times [0, c]), \) the sequence \((u_n)_{n \geq 0}\) defined by:

\[
u_n(x, t, a, b) = f(x, t) + \int_0^t \int_a^b K(x, t, y, s, u_{n-1}(y, s, a, b)) dyds
\]

converges uniformly to \( u^* \), \( \forall (x, t, a, b) \in [\alpha, \beta] \times [0, c] \times [\alpha, \beta] \times [\alpha, \beta]. \)

c) The function \( u^*, (x, t, a, b) \mapsto u^*(x, t, a, b) \) is continuous: \( u^* \in C([\alpha, \beta] \times [0, c] \times [\alpha, \beta] \times [\alpha, \beta]); \)

d) If \( K(x, t, y, \cdot) \in C^1(\mathbb{R}), \forall (x, t, y, s) \in [\alpha, \beta] \times [0, c] \times [\alpha, \beta] \times [0, c], \) then \( u^*(x, t, \cdot, \cdot) \in C^1([\alpha, \beta] \times [\alpha, \beta]), \forall (x, t) \in [\alpha, \beta] \times [0, c]. \)

Proof. Let the space \( C([a, b] \times [0, c], \mathbb{R}) \) be endowed with a suitable norm,

\[
\| u \|_{BC} := \sup \{ \| u(x, t) \| e^{-\tau t} : x \in [a, b], t \in [0, c] \}, \quad \tau > 0
\]

Let \( X := C([\alpha, \beta] \times [0, c] \times [\alpha, \beta] \times [\alpha, \beta]). \) We consider the operator \( B : X \to X \) defined by:

\[
B(u)(x, t, a, b) := f(x, t) + \int_0^t \int_a^b K(x, t, y, s, u(y, s, a, b)) dyds
\]

From (ii), applying the Contraction Principle, it follows that \( B \) is a contraction, so we have a), b) and c).

For all \( a < b \in [\alpha, \beta], \) there is a unique solution \( u^*(\cdot, \cdot, a, b) \in C([\alpha, \beta] \times [0, c]), \) so we have:

\[
u^*(x, t, a, b) = f(x, t) + \int_0^t \int_a^b K(x, t, y, s, u^*(y, s, a, b)) dyds
\]

Let us prove that \( \frac{\partial u^*(x, t, a, b)}{\partial a} \) and \( \frac{\partial u^*(x, t, a, b)}{\partial b} \) exist and they are continuous.

1. Supposing that \( \frac{\partial u^*(x, t, a, b)}{\partial a} \) exists, from (4) we obtain:

\[
\frac{\partial u^*(x, t, a, b)}{\partial a} = -\int_0^t K(x, t, a, s, u^*(a, s, a, b)) ds + \\
\int_0^t \int_a^b \frac{\partial K(x, t, y, s, u^*(y, s, a, b))}{\partial a} \cdot \frac{\partial u^*(y, s, a, b)}{\partial a} dyds
\]
This relationship suggest us to consider the next operator:

\[ C : X \times X \to X, \text{ defined by:} \]

\[ C(u, v)(x, t, a, b) := - \int_0^t K(x, t, a, s, u(a, s, a, b))ds + \]

\[ + \int_0^t \int_a^b \partial K(x, t, y, s, u(y, s, a, b)) \partial a \cdot v(y, s, a, b)dyds \]

Let \( u^* \) be the unique fixed point of \( B \). The operator \( C(u, \cdot) \) is a contraction for all \( u \in X \) and let \( v^* \) be the unique fixed point of \( C(u^*, \cdot) \).

If we define the operator \( A : X \times X \to X \times X \),

\[ A(u, v)(x, t, a, b) := (B(u)(x, t, a, b), C(u, v)(x, t, a, b)) \]

then the conditions of the Theorem 2.1 are fulfilled. It follows that \( A \) is a Picard operator and \( F_A = \{(u^*, v^*)\} \).

Consider the sequences \((u_n)_{n \geq 0}\) and \((v_n)_{n \geq 0}\) defined by:

\[ u_n(x, t, a, b) := B(u_{n-1}(x, t, a, b)) = \]

\[ = f(x, t) + \int_0^t \int_a^b K(x, t, y, s, u_{n-1}(y, s, a, b))dyds \quad \forall n \geq 1 \]

\[ v_n(x, t, a, b) := C(u_{n-1}(x, t, a, b), v_{n-1}(x, t, a, b)) = \]

\[ = - \int_0^t K(x, t, a, s, u_{n-1}(a, s, a, b))ds + \]

\[ + \int_0^t \int_a^b \partial K(x, t, y, s, u_{n-1}(y, s, a, b)) \partial a \cdot v_{n-1}(y, s, a, b)dyds \quad \forall n \geq 1 \]

We have:

\[ u_n \Rightarrow u^* \quad (n \to \infty), \quad v_n \Rightarrow v^* \quad (n \to \infty) \tag{5} \]

uniformly for \((x, t, a, b) \in [\alpha, \beta] \times [0, c] \times [\alpha, \beta] \times [\alpha, \beta] \).

We take \( u_0 = v_0 := 0 \), so \( v_1 = \frac{\partial u_1}{\partial a} \).

By induction we can prove that \( v_n = \frac{\partial u_n}{\partial a} \) for all \( n \) and from (5) results:

\[ \frac{\partial u_n}{\partial a} \Rightarrow v^* \quad (n \to \infty) \]
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Using a Weierstrass theorem, it follows that \( \frac{\partial u^*}{\partial a} \) exists and

\[
\frac{\partial u^*(x, t, a, b)}{\partial a} = v^*(x, t, a, b).
\]

2. By a similar way, we can prove the existence and the continuity of \( \frac{\partial u^*}{\partial b} \).

\[
\square
\]

Remark 3.1. We can also consider the following integral equation of Volterra-Fredholm type:

\[
\begin{align*}
  u(x, t) &= f(x, t) + \int_0^t \int_a^b K(x, t, y, s, u(y, s), \lambda) dy ds \\
  &\quad \forall t \in [0, c], \forall x \in [a, b], \text{where } \lambda \in \mathbb{R} \text{ and we can prove the differentiability of the solution with respect to the parameter } \lambda.
\end{align*}
\]

This case will be presented elsewhere.

References


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