AN INTEGRAL EQUATION WITH MODIFIED ARGUMENT

MARIA DOBRITŞOIU

Abstract. By the fixed point theorem given in the first part of Rus [3] and an idea of Sotomayor [9], a theorem of differentiability of the solution of the equation
\[ x(t) = \int_{a}^{b} K(t, s, x(s), x(\varphi(s)))ds + g(t), \quad t \in [\alpha, \beta] \]
is given.

1. Notations and preliminaries

Let \( X \) be a nonempty set, \( A : X \rightarrow X \) an operator and we shall use the following notation:

\[ F_A := \{ x \in X \mid A(x) = x \} \] - the fixed point set of \( A \).

Definition 1.1. (Rus [6] or [7]) Let \((X, d)\) be a metric space. An operator \( A : X \rightarrow X \) is Picard operator if there exists \( x^* \in X \) such that:

(a) \( F_A = \{ x^* \} \)

(b) the sequence \( (A^n(x_0))_{n \in \mathbb{N}} \) converges to \( x^* \), for all \( x_0 \in X \).

Definition 1.2. (Rus [6] or [7]) Let \((X, d)\) be a metric space. An operator \( A : X \rightarrow X \) is weakly Picard operator if the sequence \( (A^n(x_0))_{n \in \mathbb{N}} \) converges for all \( x_0 \in X \) and the limit (which may depend on \( x_0 \)) is a fixed point of \( A \).

If \( A \) is a weakly Picard operator, then we consider the following operator

\[ A^\infty : X \rightarrow X, \quad A^\infty(x) = \lim_{n \rightarrow \infty} A^n(x) \]
It is clear that $A^\infty(X) = FA$.

In the section 2 we need the following results (see [4] and [3]).

**Perov’s theorem.** Let $(X,d)$, with $d(x,y) \in \mathbb{R}^m$, be a complete generalized metric space and $A : X \to X$ an operator. We suppose that there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$, such that

(i) $d(A(x), A(y)) \leq Qd(x,y)$, for all $x, y \in X$;

(ii) $Q \to 0$ as $n \to \infty$.

Then

(a) $F_A = \{x^*\}$,

(b) $A^n(x) \to x^*$ as $n \to \infty$ and

$$d(A^n(x), x^*) \leq (I - Q)^{-1}Q^{n}d(x_0, A(x_0)).$$

**Rus theorem.** (Rus [3]) Let $(X,d)$ be a metric space (generalized or not) and $(Y, \rho)$ be a complete generalized metric space ($\rho(x,y) \in \mathbb{R}^m$).

Let $A : X \times Y \to X \times Y$ be a continuous operator. We suppose that:

(i) $A(x,y) = (B(x), C(x,y))$, for all $x \in X, y \in Y$;

(ii) $B : X \to X$ is a weakly Picard operator;

(iii) There exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$, $Q^n \to 0$ as $n \to \infty$, such that

$$\rho(C(x,y_1), C(x,y_2)) \leq Q\rho(y_1, y_2),$$

for all $x \in X, y_1$ and $y_2 \in Y$.

Then the operator $A$ is weakly Picard operator. Moreover, if $B$ is Picard operator, then $A$ is Picard operator.

In the section 3 we need the following definition and result (see [8]).

**Definition 1.3.** (Rus [8]) A matrix $Q \in M_{nn}(\mathbb{R})$ converges to zero if $Q^k$ converges to the zero matrix as $k \to \infty$.

**Theorem 1.1.** (Rus [8]) Let $Q \in M_{nn}(\mathbb{R}_+)$. The following statements are equivalent:

(i) $Q^k \to 0$ as $k \to \infty$;
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(ii) The eigenvalues $\lambda_k, k = 1, n$ of the matrix $Q$, verify the condition
$|\lambda_k| < 1, k = 1, n$;

(iii) The matrix $I - Q$ is non-singular and $(I - Q)^{-1} = I + Q + \cdots + Q^n + \ldots$.

2. The main result

We consider the following Fredholm integral equation with modified argument

$$x(t) = \int_a^b K(t, s, x(s), x(\varphi(s)))ds + g(t), \quad t \in [\alpha, \beta],$$

where $\alpha, \beta \in R, \alpha \leq \beta, a, b \in [\alpha, \beta], g \in C([\alpha, \beta], R^m)$, $K \in C([\alpha, \beta] \times [\alpha, \beta] \times R^m \times R^m, R^m)$, $x \in C([\alpha, \beta], R^m)$ and $\varphi \in C([\alpha, \beta], [\alpha, \beta])$.

We have

**Theorem 2.1.** We suppose that there exists $Q \in M_{mm}(R_+)$ such that:

(i) $[(\beta - \alpha)Q]^n \to 0$ as $n \to \infty$;

(ii) \[
\begin{pmatrix}
|K_1(t, s, u, v) - K_1(t, s, w, z)| \\
\vdots \\
|K_m(t, s, u, v) - K_m(t, s, w, z)|
\end{pmatrix}
\leq Q
\begin{pmatrix}
|u_1 - w_1| + |v_1 - z_1| \\
\vdots \\
|u_m - w_m| + |v_m - z_m|
\end{pmatrix}
\]

for all $u, v, w, z \in R^m$, $t, s \in [\alpha, \beta]$.

Then

(a) the equation (1) has in $C([\alpha, \beta], R^m)$ a unique solution, $x^*(\cdot, a, b)$;

(b) for all $x^0 \in C([\alpha, \beta], R^m)$ the sequence $(x^n)_{n \in \mathbb{N}}$, defined by

$$x^{n+1}(t; a, b) := \int_a^b K(t, s, x^n(s; a, b), x^n(\varphi(s); a, b))ds + g(t)$$

converges uniformly to $x^*$, for all $t, a, b \in [\alpha, \beta]$, and

$$\begin{pmatrix}
|x_1^n(t; a, b) - x_1^1(t; a, b)| \\
\vdots \\
|x_m^n(t; a, b) - x_m^1(t; a, b)|
\end{pmatrix}
\leq [I - (\beta - \alpha)Q]^{-1}[(\beta - \alpha)Q]^n
\begin{pmatrix}
|x_1^0(t; a, b) - x_1^1(t; a, b)| \\
\vdots \\
|x_m^0(t; a, b) - x_m^1(t; a, b)|
\end{pmatrix}$$

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(c) the function

\[ x^* : [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}^m, \quad (t, a, b) \to x^*(t; a, b) \]

is continuous;

(d) if \( K(t, s, \cdot, \cdot) \in C^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m) \), for all \( t, s \in [\alpha, \beta] \), then

\[ x^*(t; \cdot, \cdot) \in C^1([\alpha, \beta] \times [\alpha, \beta], \mathbb{R}^m), \text{ for all } t \in [\alpha, \beta]. \]

Proof. Let \( \| \cdot \| \) be a generalized Chebyshev norm on

\[ X := C([\alpha, \beta]^3, \mathbb{R}^m) \]

i.e.

\[ \| x \| := \left( \| x_1 \|_\infty \ldots \| x_m \|_\infty \right). \]

Let we consider the operator \( B : X \to X \) defined by

\[ B(x)(t; a, b) := \int_a^b K(t, s, x(s; a, b), x(\varphi(s); a, b))ds \]

for all \( t, a, b \in [\alpha, \beta] \).

From (i) and (ii) and the Perov’s theorem we have (a)+(b)+(c).

(d) Let we prove that there exists \( \frac{\partial x^*}{\partial a} \) and \( \frac{\partial x^*}{\partial a} \in X \).

If we suppose that there exists \( \frac{\partial x^*}{\partial a} \), then from (1) we have

\[
\frac{\partial x^*}{\partial a} (t; a, b) = -K(t, a, x^*(a; a, b), x^*(\varphi(a); a, b)) + \\
+ \int_a^b \left[ \left( \frac{\partial K}{\partial x_i}(t, s; x^*(s; a, b), x^*(\varphi(s); a, b)) \right) \frac{\partial x^*(s; a, b)}{\partial a} + \\
+ \left( \frac{\partial K}{\partial x_i}(t, s; x^*(s; a, b), x^*(\varphi(s); a, b)) \right) \frac{\partial x^*(\varphi(s); a, b)}{\partial a} \right] ds.
\]

This relation suggest to consider the following operator

\[ C : X \times X \to X, \]
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\[ C(x, y)(t; a, b) := -K(t, a, x(a; b), x(\varphi(a); a, b)) + \]
\[ + \int_a^b \left[ \left( \frac{\partial K_j(t, s, x(s; a, b), x(\varphi(s); a, b))}{\partial x_i} \right) y(s; a, b) \right. \]
\[ + \left. \left( \frac{\partial K_j(t, s, x(s; a, b), x(\varphi(s); a, b))}{\partial x_i} \right) y(\varphi(s); a, b) \right] ds. \]

From (ii), we remark that
\[ \left( \left| \frac{\partial K_j(t, s, u, v)}{\partial x_i} \right| \right) \leq Q \]  
for all \( t, s \in [\alpha, \beta] \) and \( u, v \in \mathbb{R}^m \).

From (2) and (3) it follows that
\[ \| C(x, y_1) - C(x, y_2) \| \leq (\beta - \alpha)Q, \]
for all \( x, y_1, y_2 \in X \).

If we take the operator
\[ A : X \times X \to X \times X, \quad A = (B, C), \]
then we are in the conditions of the Rus theorem. From this theorem, the operator \( A \) is a Picard operator and the sequences

\[ x^{n+1}(t; a, b) = \int_a^b K(t, s, x^n(s; a, b), x^n(\varphi(s); a, b))ds + g(t) \]
\[ y^{n+1}(t; a, b) := -K(t, a, x^n(a; b), x^n(\varphi(a); a, b)) + \]
\[ + \int_a^b \left[ \left( \frac{\partial K_j(t, s, x^n(s; a, b), x^n(\varphi(s); a, b))}{\partial x_i} \right) y^n(s; a, b) \right. \]
\[ + \left. \left( \frac{\partial K_j(t, s, x^n(s; a, b), x^n(\varphi(s); a, b))}{\partial x_i} \right) y^n(\varphi(s); a, b) \right] ds \]
converges uniformly (with respect to \( t, a, b \in [\alpha, \beta] \)) to \((x^*, y^*) \in FA\), for all \( x^0, y^0 \in X \).

If we take \( x^0 = y^0 = 0 \), then \( y^1 = \frac{\partial x^1}{\partial a} \). By induction we prove that
\[ y^n = \frac{\partial x^n}{\partial a}. \]
Thus
\[ x^n \xrightarrow{n \to \infty} x^* \] as \( n \to \infty \).
These imply that there exists \( \frac{\partial x^*}{\partial a} \) and \( \frac{\partial x^*}{\partial b} = y^* \).

By a similar way we prove that there exists \( \frac{\partial x^*}{\partial b} \). \( \Box \)

3. Example

In what follows we consider the following system of Fredholm integral equations

\[
\begin{align*}
x_1(t) &= \int_a^b \left[ \frac{1}{8} (t + s) x_1(s) + \frac{1}{4} x_1(s/2) \right] ds + 1 - \cos t \\
x_2(t) &= \int_a^b \left[ \frac{1}{2} x_1(s) + \frac{2t + s}{4} x_2(s) + \frac{3}{4} x_2(s/2) \right] ds + \sin t,
\end{align*}
\]

(4)

\( t, a, b \in [0, 1] \), where \( a, b \in [0, 1] \), \( g \in C([0, 1], \mathbb{R}^2) \), \( g(t) = (g_1(t), g_2(t)) \), \( g_1(t) = 1 - \cos t \), \( g_2(t) = \sin t \), \( K \in C([0, 1] \times [0, 1] \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2) \),

\[
K(t, s, x(s), x(\varphi(s))) = (K_1(t, s, x(s), x(\varphi(s))), K_2(t, s, x(s), x(\varphi(s)))),
\]

\[
K_1 = \frac{1}{8} (t + s) x_1(s) + \frac{1}{4} x_1(s/2), \quad K_2 = \frac{1}{2} x_1(s) + \frac{2t + s}{4} x_2(s) + \frac{3}{4} x_2(s/2),
\]

\( \varphi \in C([0, 1], [0, 1]) \), \( \varphi(s) = s/2 \) and \( x \in C([0, 1], \mathbb{R}^2) \).

From the condition (ii) of the theorem 2.1 we have

\[
\left( \begin{array}{c} |K_1(t, s, x(s), x(s/2)) - K_1(t, s, x(s), z(s/2))| \\ |K_2(t, s, x(s), x(s/2)) - K_2(t, s, x(s), z(s/2))| \end{array} \right) \leq \left( \begin{array}{cc} 1/4 & 0 \\ 1/2 & 3/4 \end{array} \right) \left( \begin{array}{c} |x_1(s) - z_1(s)| + |x_1(s/2) - z_1(s/2)| \\ |x_2(s) - z_2(s)| + |x_2(s/2) - z_2(s/2)| \end{array} \right), \quad t, s \in [0, 1],
\]

which lead to matrix

\[
Q = \left( \begin{array}{cc} 1/4 & 0 \\ 1/2 & 3/4 \end{array} \right), \quad Q \in M_{22}(\mathbb{R}_+),
\]

that according to the theorem 1.1 and definition 1.3, converges to zero.

Therefore the conditions of the theorem 2.1 are satisfies and we have

- the system of equations (4) has in \( C([0, 1], \mathbb{R}^2) \) a unique solution \( x^*(\cdot, a, b) \);
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- for all \( x^0 \in C([0, 1], \mathbb{R}^2) \) the sequence \( (x^n)_{n \in \mathbb{N}} \), defined by
  \[
  x^{n+1}(t; a, b) := \int_a^b K(t, s, x^n(s; a, b), x^n(\varphi(s); a, b))ds + g(t)
  \]
converges uniformly to \( x^* \), for all \( t, a, b \in [0, 1] \), and

\[
\begin{pmatrix}
|x_1^n(t; a, b) - x_1^1(t; a, b)| \\
\vdots \\
|x_m^n(t; a, b) - x_m^1(t; a, b)|
\end{pmatrix} \leq [I - Q]^{-1} Q^n
\begin{pmatrix}
|x_1^n(t; a, b) - x_1^1(t; a, b)| \\
\vdots \\
|x_m^n(t; a, b) - x_m^1(t; a, b)|
\end{pmatrix}
\]

- the function
  \[
x^* : [0, 1] \times [0, 1] \times [0, 1] \to \mathbb{R}^2, \quad (t; a, b) \to x^*(t; a, b)
  \]
is continuous;

- if \( K(t, s, \cdot, \cdot) \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2) \), for all \( t, s \in [0, 1] \), then
  \( x^*(t; \cdot, \cdot) \in C^1([0, 1] \times [0, 1], \mathbb{R}^2) \), for all \( t \in [0, 1] \).

References


Faculty of Science, University of Petrosani, Petrosani, Romania
E-mail address: mariadobritoiu@yahoo.com

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