THE BETA APPROXIMATING OPERATORS OF SECOND KIND

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Abstract. We shall define a general linear transform from which we obtain as particular case the beta second kind transform:

\[ T_{p,q}f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1 + u)^{p+q}} f(u) \, du \]  

(\ast)

We consider here only the particular case \( a = 1 \).

We obtain several positive linear operators as a particular case of this beta second kind transform. We apply the transform (\ast) to Baskakov’s operator and Bleimann, Butzer and Hahn operator respectively and we obtain new generalization of these operators.

1. Introduction

Many authors introduced and studied positive linear operators, using Euler’s beta function of second kind: [1], [2], [5], [6], [7], [9].

Euler’s beta function of second kind is defined for \( p > 0, q > 0 \) by the following formula

\[ B(p,q) = \int_0^\infty \frac{u^{p-1}}{(1 + u)^{p+q}} \, du \]  

(1.1)

The beta transform of the function \( f \) is defined by the following formula

\[ B_{p,q}f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1 + u)^{p+q}} f(u) \, du \]

We shall define a more general linear transform from which we obtain as particular case the beta second-kind transform.

For \( a, b \in \mathbb{R} \) we define the \( (a,b) \)-beta transform of a function \( f \)

\[ B_{p,q}^{(a,b)}f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1 + u)^{p+q}} f \left( \frac{u^a}{(1 + u)^{a+b}} \right) \, du \]  

(1.2)
where \( B(\cdot, \cdot) \) is the beta function (1.1) and \( f \) is any real measurable function defined on \((0, \infty)\) such that \( B_{p,q}^{(a,b)} |f| < \infty \).

2. The beta second-kind transform. Case \( a = 1 \)

Let us denote by \( M[0, \infty) \) the linear space of functions defined for \( t \geq 0 \), bounded and Lebesgue measurable in each interval \([c, d]\), where \( 0 < c < d < \infty \).

If we consider in (1.2) \( a + b = 0 \) we obtain the second-kind transform of function \( f \in M[0, \infty) \)

\[
T^{(a)}_{p,q} f = B^{(a,-a)}_{p,q} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u^a) du
\]  

(2.1)

such that \( T^{(a)}_{p,q} |f| < \infty \). Clearly \( T^{(a)}_{p,q} \) is a positive linear functional.

We shall consider here only the particular case \( a = 1 \) (see also [9])

\[
T_{p,q} f = T^{(1)}_{p,q} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u) du
\]  

(2.2)

for \( f \in M[0, \infty) \) such that \( T_{p,q} |f| < \infty \).

Remark. If \( a = -1 \) we obtain

\[
T^{(-1)}_{p,q} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{1}{u}\right) du
\]

Denoting \( u = v^{-1} \) we can write

\[
T^{(-1)}_{p,q} f = \frac{1}{B(p,q)} \int_0^\infty \frac{v^{p-1}}{(1+v)^{p+q}} f(v) \frac{1}{v^2} dv =
\]

\[
= \frac{1}{B(p,q)} \int_0^\infty \frac{v^{p+q}}{v^{p+1} (1+v)^{p+q}} f(v) dv =
\]

\[
= \frac{1}{B(p,q)} \int_0^\infty \frac{v^{q-1}}{(v+1)^{p+q}} f(v) dv = T^{(1)}_{p,q} f.
\]

That is \( T^{(-1)}_{p,q} f = T^{(1)}_{p,q} f = T_{p,q} f \).

Lemma 2.1. [9] The moment of order \( k \) \((1 \leq k < q)\) of the functional \( T_{p,q} \) has the following value

\[
T_{p,q} e_k = \frac{p(p+1) \ldots (p+k-1)}{(q-1) \ldots (q-k)} , \quad 1 \leq k < q
\]  

(2.3)
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We impose that $T_{p,q}e_1 = e_1$, that is $p = (q - 1)x$, $q > 1$ and we obtain

$$(T_q f)(x) = \frac{1}{B((q-1)x,q)} \int_0^\infty \frac{u^{(q-1)x-1}}{(1+u)^{q-1}(x+1)+1} f(u)du$$ (2.4)

Lemma 2.2. One has

$$(T_q e_2)(x) = x^2 + \frac{x(x+1)}{q-2}, \quad q > 2$$

$$(T_q((t-x)^2);x) = \frac{x(x+1)}{q-2}, \quad q > 2. \quad (2.5)$$

Proof. It is obtained from Lemma 2.1 for $p = (q - 1)x$. □

Particular cases

a) If in (2.4) we choose $q = 1 + \frac{1}{\alpha}$, $\alpha \in (0,1)$, then we give the positive linear operator $L_\alpha$ defined for $\alpha \in (0,1)$ and $x \geq 0$:

$$(L_\alpha f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^\infty \frac{u^{\frac{x}{\alpha}-1}}{(1+u)^{\frac{1}{\alpha}+1}} f(u)du$$ (2.6)

considered in [9] (see also [1], [2], [7]).

Lemma 2.3. One has

$$(L_\alpha e_2)(x) = x^2 + \frac{\alpha}{1-\alpha} x(1+x).$$

$$L_\alpha((t-x)^2);x) = \frac{\alpha}{1-\alpha} x(1+x).$$

Proof. We take $q = (\alpha + 1)/\alpha$ in (2.5). □

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$L_{1/n}((t-x)^2);x) = \frac{x(1+x)}{n-1}.$$ 

b) If we choose in (2.4) $q = \frac{1}{\alpha(1+x)} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \frac{1}{\alpha} - 1\right)$, we obtain the beta type operator $H_\alpha$, given by

$$(H_\alpha f)(x) = \frac{1}{B\left(\frac{x}{\alpha(1+x)}, \frac{1}{\alpha(1+x)} + 1\right)} \int_0^\infty \frac{u^{\frac{x}{\alpha(1+x)}-1}}{(1+u)^{\frac{1}{\alpha(1+x)}+1}} f(u)du$$ (2.7)
where \( f \in M[0, \infty) \) such that \( H_\alpha |f| < \infty \), considered by J. Adell [2].

**Lemma 2.4.** One has

\[
(H_\alpha e_2)(x) = x^2 + \frac{\alpha x(1 + x)^2}{1 - \alpha(x + 1)}
\]

\[
H_\alpha((t - x)^2; x) = \frac{\alpha x(1 + x)^2}{1 - \alpha(x + 1)}
\]

**Proof.** We take \( q = \frac{1}{\alpha(x + 1)} + 1 \) in Lemma 2.2. \( \square \)

For \( \alpha = 1/n, n \in \mathbb{N} \), we obtain

\[
H_{1/n}((t - x)^2; x) = \frac{x(1 + x)^2}{n - 1 - x}.
\]

c) If we put in (2.4) \( q = 1 + \frac{1}{\alpha x}, \alpha \in (0, 1), x \in \left( 0, \frac{1}{\alpha} \right) \), we obtain the positive linear operator \( M_{\alpha} \), given by

\[
(M_{\alpha} f)(x) = \frac{1}{B \left( \frac{1}{\alpha}, \frac{1}{\alpha x} + 1 \right)} \int_0^\infty \frac{u^{\frac{1}{\alpha} - 1}(1 + u)^{1 + \frac{1}{\alpha x}}}{\alpha(1 + u)^{\frac{1}{\alpha} + 1}} f(u) du \quad (2.8)
\]

where \( f \in M(0, \infty) \) such that \( M_{\alpha}|f| < \infty \).

**Lemma 2.5.** One has

\[
(M_{\alpha} e_2)(x) = x^2 + \frac{\alpha x^2(x + 1)}{1 - \alpha x}
\]

\[
M_{\alpha}((t - x)^2; x) = \frac{\alpha x^2(x + 1)}{1 - \alpha x}
\]

**Proof.** The above identities are implied by Lemma 2.2. \( \square \)

For \( \alpha = 1/n, n \in \mathbb{N} \), we obtain

\[
M_{1/n}((t - x)^2; x) = \frac{x^2(x + 1)}{n - x}.
\]

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3. Generalized Baskakov operator

Let be $B_n$ the Baskakov operator [3]

$$
(B_nf)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right) \quad (3.1)
$$

Now let us apply the transform $T_{p,q}$ (2.2) to Baskakov’s operator (3.1) and we obtain (see [9])

**Theorem 3.1.** The $T_{p,q}$ transform of $B_n f$ can be expressed by the following form

$$
T_{p,q}(B_n f) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(p)_k(q)_n}{(p+q)_{n+k}} f\left(\frac{k}{n}\right) \quad (3.2)
$$

where $(a)_m := a(a+1)\ldots(a+m-1)$.

**Proof.**

$$
T_{p,q}(B_n f) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{u^k}{(1+u)^{n+k}} f\left(\frac{k}{n}\right) du =
$$

$$
= \frac{1}{B(p,q)} \sum_{k=0}^{\infty} \binom{n+k-1}{k} f\left(\frac{k}{n}\right) \int_0^\infty \frac{u^{p+k-1}}{(1+u)^{p+q+n+k}} du =
$$

$$
= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{B(p+k,q+n)}{B(p,q)} f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(p)_k(q)_n}{(p+q)_{n+k}} f\left(\frac{k}{n}\right). \Box
$$

**Theorem 3.2.** One has

$$
T_{p,q}(B_n e_1) = T_{p,q}(B_n e_1) = \frac{p}{q-1} \quad (3.3)
$$

$$
T_{p,q}(B_n e_2) = T_{p,q}(B_n e_2) = \frac{p(p+1)}{(q-2)(q-1)} + \frac{1}{n} \frac{p(p+q-1)}{(q-2)(q-1)}.
$$

**Proof.**

$$
T_{p,q}(B_n e_1) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} u du
$$

$$
= \frac{1}{B(p,q)} \int_0^\infty \frac{u^p}{(1+u)^{p+q}} du = B(p+1,q-1) \frac{1}{B(p,q)} = \frac{p}{q-1}.
$$

$$
T_{p,q}(B_n e_2) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} \left(u^2 + \frac{u(a+1)}{n}\right) du =
$$

$$
= \frac{1}{B(p,q)} \left( \int_0^\infty \frac{u^{p+1}}{(1+u)^{p+q}} du + \frac{1}{n} \int_0^\infty \frac{u^p}{(1+u)^{p+q-1}} du \right) =
$$
\begin{align*}
&= \frac{1}{B(p, q)} \left( B(p + 2, q - 2) + \frac{1}{n} B(p + 1, q - 2) \right) = \\
&= \frac{B(p + 2, q - 2)}{B(p, q)} + \frac{1}{n} \frac{B(p + 1, q - 2)}{B(p, q)} = \frac{p(p + 1)}{(q - 2)(q - 1)} + \frac{1}{n} \frac{p(p + 1)}{(q - 2)(q - 1)}. \square
\end{align*}

We impose that $T_n^{(p, q)} e_1 = e_1$, that is $p = (q - 1)x$, $x > 0$, $q > 2$. We obtain from Theorem 3.1 and Theorem 3.2

**Corollary 3.3.** One has

$$
T_n^{(q)} f = T_q(B_n f) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{((q-1)x)k(q)_n}{(q-1)x q^n+k} f \left( \frac{k}{n} \right)
$$

(3.4)

**Corollary 3.4.** One has

\begin{align*}
(T_n^{(q)} e_1)(x) &= x, \quad (T_n^{(q)} e_2)(x) = x^2 + \frac{x(1+x)}{q-2} \left( 1 + \frac{q-1}{n} \right) \\
T_n^{(q)} ((t-x)^2; x) &= \frac{x(1+x)}{q-2} \left( 1 + \frac{q-1}{n} \right)
\end{align*}

(3.5)

**Proof.** Choosing $p = (q - 1)x$ in Theorem 3.2, the conclusion follows. \square

**Particular cases**

a) If we put in (3.4) $q = \frac{1}{\alpha} + 1$, $\alpha \in (0, 1)$, we obtain the operator considered by D. D. Stancu [8], as a generalization of the Baskakov operator

$$
\mathcal{L}_n^{(\alpha)}(x) = L_\alpha(B_n f)(x) = \sum_{k=0}^{\infty} I_{n,k}(x, \alpha) f \left( \frac{k}{n} \right)
$$

(3.6)

where

$$
I_{n,k}(x, \alpha) = \binom{n+k-1}{k} x(x+\alpha) \cdots (x+(k-1)\alpha)(1+\alpha)(1+2\alpha) \cdots (1+n\alpha)
$$

\(1+x+\alpha)(1+x+2\alpha) \cdots (1+x+(n+k)\alpha)

**Corollary 3.5.** One has

$$
\mathcal{L}_n^{(\alpha)}((t-x)^2; x) = \frac{\alpha(1+x)}{1-\alpha} \left( 1 + \frac{1}{n\alpha} \right)
$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$
\mathcal{L}_n((t-x)^2; x) = \frac{2x(1+x)}{n-1}.
$$
b) For \( q = \frac{1}{\alpha(x+1)} + 1, \alpha \in (0,1), x \in \left(0, \frac{1}{\alpha} - 1\right) \), we obtain by (3.4) a new generalization of the Baskakov operator

\[
(H_n^{(\alpha)}) f(x) = H_\alpha(B_n f)(x) = \sum_{k=0}^{\infty} h_{n,k}(x, \alpha) f\left(\frac{k}{n}\right)
\]

where

\[
h_{n,k}(x, \alpha) = \binom{n+k-1}{k} \frac{x(x+\alpha(1+x)) \ldots (x+(k-1)\alpha(x+1))(1+\alpha(1+x)) \ldots (1+n\alpha(1+x))}{(1+x)^{n+k}(1+\alpha) \ldots (1+(n+k)\alpha)}
\]

**Corollary 3.6.** One has

\[
H_n^{(\alpha)}((t-x)^2; x) = \frac{\alpha x(1+x)^2}{1-\alpha(1+x)} \left(1 + \frac{1}{\alpha n(1+x)}\right)
\]

For \( \alpha = 1/n, n \in \mathbb{N} \), we obtain

\[
H_n((t-x)^2; x) = \frac{x(x+1)(x+2)}{n-1-x}.
\]

c) If we put in (3.4) \( q = 1 + \frac{1}{\alpha x}, \alpha \in (0,1), x \in \left(0, \frac{1}{\alpha}\right) \), we obtain a new generalization of the Baskakov operator

\[
(M_n^{(\alpha)}) f(x) = M_\alpha(B_n f)(x) = \sum_{k=0}^{\infty} m_{n,k}(x, \alpha) f\left(\frac{k}{n}\right)
\]

where

\[
m_{n,k}(x, \alpha) = \binom{n+k-1}{k} \frac{(1+\alpha) \ldots (1+(k-1)\alpha)(1+\alpha x) \ldots (1+n\alpha x)}{(x+1+\alpha x) \ldots (x+1+(n+k)\alpha x)} x^k
\]

**Corollary 3.7.** One has

\[
M_n^{(\alpha)}((t-x)^2; x) = \frac{\alpha x^2(x+1)}{1-\alpha x} \left(1 + \frac{1}{\alpha nx}\right)
\]

For \( \alpha = 1/n, n \in \mathbb{N} \), we obtain

\[
M_n((t-x)^2; x) = \frac{x(x+1)^2}{n-x}.
\]
4. Generalized Bleimann, Butzer, Hahn operator

Let be \( \tilde{B}_n \) the Bleimann, Butzer, Hahn operator \([4]\)

\[
(\tilde{B}_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} \cdot f \left( \frac{k}{n-k+1} \right) \tag{4.1}
\]

Now let us apply the transform \( T_{p,q} (2.2) \) to Bleimann, Butzer, Hahn’s operator (4.1) and we obtain

**Theorem 4.1.** The \( T_{p,q} \) transform of \( \tilde{B}_n f \) can be expressed by the following form

\[
\tilde{T}_{n}^{(p,q)} f = T_{p,q}(\tilde{B}_n f) = \sum_{k=0}^{n} \binom{n}{k} \frac{(p)k(q)n-k}{(p+q)n} f \left( \frac{k}{n-k+1} \right) \tag{4.2}
\]

**Proof.**

\[
\tilde{T}_{n}^{(p,q)} f = T_{p,q}(\tilde{B}_n f) = \frac{1}{B(p,q)} \int_0^{\infty} u^{p-1} \left( 1+u \right)^{p+q} \sum_{k=0}^{n} \binom{n}{k} \frac{u^k}{(1+u)^n} f \left( \frac{k}{n-k+1} \right) du =
\]

\[
= \frac{1}{B(p,q)} \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{k}{n-k+1} \right) \int_0^{\infty} u^{p+k-1} \left( 1+u \right)^{p+q+n} du =
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \frac{B(p+k,q+n-k)}{B(p,q)} f \left( \frac{k}{n-k+1} \right) =
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \frac{(p)k(q)n-k}{(p+q)n} f \left( \frac{k}{n-k+1} \right). \quad \Box
\]

**Particular cases**

a) If we put in (4.2), \( p = \frac{x}{\alpha} \), \( q = \frac{1}{\alpha} + 1 \), \( \alpha \in (0,1) \), \( x \geq 0 \), we obtain the operator introduced by J. Adell [2] as a generalization of the Bleimann, Butzer, Hahn operator

\[
(L_n^{(\alpha)} f)(x) = L_{\alpha}(\tilde{B}_n f)(x) = \sum_{k=0}^{n} \tilde{l}_{n,k}(x,\alpha) f \left( \frac{k}{n-k+1} \right) \tag{4.3}
\]

where

\[
\tilde{l}_{n,k}(x,\alpha) = \binom{n}{k} x(x+\alpha) \ldots (x+(k-1)\alpha)(1+\alpha) \ldots (1+(n-k)\alpha) \frac{1}{(x+1+\alpha) \ldots (x+1+n\alpha)}
\]
b) For \( p = \frac{x}{\alpha(1+x)} \) and \( q = \frac{1}{\alpha(1+x)} + 1, \alpha \in (0,1), x \in \left(0, \frac{1}{\alpha} - 1\right) \), we obtain by (4.2) a new generalization of the Bleimann, Butzer, Hahn operator
\[
(\tilde{H}_n^{(\alpha)} f)(x) = H_\alpha(\tilde{B}_n f)(x) \sum_{k=0}^{n} \tilde{h}_{n,k}(x, \alpha) f \left( \frac{k}{n - k + 1} \right) \quad (4.4)
\]
where
\[
\tilde{h}_{n,k}(x, \alpha) = \binom{n}{k} \frac{x(x + \alpha(1+x)) \ldots (x + (k-1)\alpha(1+x))(1 + \alpha(1+x)) \ldots (1 + (n-k)\alpha(1+x))}{(1 + x)^{\alpha}(1 + 2\alpha) \ldots (1 + n\alpha)}.
\]

c) If we put in (4.2) \( p = \frac{1}{\alpha x}, q = \frac{1}{\alpha x} + 1, \alpha \in (0,1), x \in \left(0, \frac{1}{\alpha} \right) \), we obtain a new generalization of the Bleimann, Butzer, Hahn operator
\[
(\tilde{M}_n^{(\alpha)} f)(x) = M_\alpha(\tilde{B}_n f)(x) = \sum_{k=0}^{n} \tilde{m}_{n,k}(x, \alpha) f \left( \frac{k}{n - k + 1} \right) \quad (4.5)
\]
where
\[
\tilde{m}_{n,k}(x, \alpha) = \binom{n}{k} \frac{(1 + \alpha) \ldots (1 + k\alpha)(1 + \alpha x) \ldots (1 + (n-k-1)\alpha x)}{(x + 1 + \alpha x) \ldots (x + 1 + n\alpha x)} x^k.
\]

References


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