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THE PROBLEM OF B. V. GNEDENKO FOR PARTIAL SUMMATION SCHEMES ON BANACH SPACE

HO DANG PHUC

Abstract. The paper deals with the problem of B. V. Gnedenko for the partial summation scheme of random vectors taking values in a Banach space. A characterization of the limit distribution class of the scheme and some conditions for the limit distribution to be convolutions of semistable distributions are given.

1. **Introduction and notation**

Let $X_1, X_2, ...$ be a sequence of independent random variables and

$$Y_n \quad = \quad a_n \sum_{i=1}^{n} X_i \; + \; x_n \quad , \quad n = 1, 2, ... \tag{1.1}$$

a sequence of normalized sums, which has a proper limit distribution Q for an appropriate choice of normalizing sequence of positive numbers (a_n) tending to 0 and of elements (x_n) from the real line. B. V. Gnedenko posed the problem of characterizing the class of the distributions $\{Q\}$ when among the distributions of the summands X_i there are only p different ones. Let this class be denoted by G_p. It is well known that G_1 coincides with the class of stable distributions. In [11] Zolotarev and Korolyuk proved that G_2 is the class of convolutions of stable distributions pairs. (This theorem is generalized to Banach valued random vectors by Jurek [5]). Further, Zinger shows that in the case of $p > 2$, the class G_p is broader than the one of stable distribution convolutions [9] and characterized it in [10].

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An extension of the stable distributions class is the one of semistable distributions, i. e. the limit distributions $\{Q\}$ when in (1.1) the index n run not over whole the sequence of natural numbers, but only along some subsequence $(k(n))$:

$$Y_n \quad = \quad a_n \sum_{i=1}^{k(n)} X_i \; + \; x_n \; , \quad n = 1, 2, ... \quad ,$$

$X_i, i = 1, 2, ...$, are independent identically distributed and k(n) tends to infinity not too fast:

$$k(n)/k(n+1) \to r \; , \quad 0 < r < 1 \; . \tag{1.2}$$

Under this idea, Chibisova [1,2] generalized the Gnedenko's problem to the partial summation scheme:

$$Y_n \; = \; a_n \left(\sum_{i=1}^{k(1,n)} X_{1,i} \; + ... + \sum_{i=1}^{k(p,n)} X_{p,i} \right) + x_n \; , \; n = 1, 2, ... \; , \tag{1.1$'$}$$

where $X_{j,i}$, $i = 1, 2, ..., k(i, j)$; $j = 1, .., p$, are independent, $X_{j,i}$, $i = 1, 2, ..., k(i, j)$, have a common distribution μ_j for $j = 1, ..., p$, and

$$1 \; > \; k(j, n)/k(j, n+1) \; \geq \; c \; , \quad j \; = \; 1, ..., p \tag{1.2$'$}$$

for some $c > 0$. The reason for taking (1.2′) instead of (1.2) to restrict the scheme (1.1′) is that for the case when $p > 1$, if the condition (1.2′) fails, the limit distribution of the scheme (1.1′) would be an arbitrary infinitely divisible distribution without normal component, although (1.2) holds for $k(n) \;\; = \;\; k(1, n) + ... + k(p, n)$ (see Theorem 1 [1]).

In this paper, we attempt to study the problem of B. V. Gnedenko for the partial summation scheme of random vectors taking values in a Banach space. A characterization of the limit distributions class of the scheme (Theorem 1) and some conditions for the limit distribution to be convolutions of semistable distributions (Theorems 2, 3 and 4) are given.

In the sequel the following notation will be used: E denotes a separable Banach space, $E^{'}$ its dual space, $< .,. >$ the dual pairing between E and $E^{'}$. Further, $P(E)$ stands for the class of all probability measures on E , $\delta(x)$ the unit mass concentrated at the point $x \in E, *$ the convolution and $\Rightarrow$ the weak convergence of measures in $P(E)$. It is well known that $P(E)$ with the weak convergence topology is a separable metric space (see [6], Theorem II.6.2). Moreover, one can find in this space a *shift-invariant metric* (e. g. the Levy-Prokhorov metric), i. e. a metric ρ such that

$$\rho(\nu * \delta(x)), \mu * \delta(x)) \quad = \quad \rho(\nu, \mu)$$

for all $x \in E,\ \mu,\ \nu \in P(E)$.

Throughout the forthcoming, unless otherwise specified, we shall denote by small italic letters x, y, z elements from $E; a, b, c, r, s, t$ positive numbers, (n) the sequence of all natural numbers and by Greek letters $\gamma, \kappa, \lambda, \mu, \nu$ measures from $P(E)$. Moreover, $(x_n), (y_n), (z_n)$, also with other subscripts or indexes, will mean sequences of elements from E. Similarly, $(a_n), (b_n), (c_n), (r(n)), (s(n)), (t(n))$ mean sequences of positive numbers, (γ_n), (κ_n), $(\lambda_n), (\mu_n), (\nu_n)$ sequences of measures from $P(E)$ and $(k(n)), (m(n))$, $(n'), (n'')$ subsequences of natural numbers.

A measure μ is called nondegenerated if it is not concentrated at any point and the power μ^{*n} is defined recursively by $\mu^{*n} = \mu^{*n-1} * \mu$. Further μ is said to be *infinitely divisible* if for every n there exists a measure μ_n such that $\mu = (\mu_n)^{*n}$. By $ID(E)$ we mean the subclass of all infinitely divisible measures from $P(E)$. Then for each $t \geq 0$ and $\mu \in ID(E)$ we can define μ^t (see [8], for example).

For a measurable map S from E to another Banach space, $S\mu$ stands for the image of μ by the map S. In particular, when S is of the form $a.I$, where I is the unit operator in E , we write straightly $a.\mu$ instead of $a.I\mu$. We say that μ belongs to *the domain of semi-attraction*, or more exactly $r-$*semi-attraction*, of λ if there exist $(a_n), (k(n))$ and (x_n) such that

$$a_n.\mu^{*k(n)} \ * \ \delta\,(x_n) \quad \Rightarrow \quad \lambda$$

and

$$k(n)/k(n+1) \ \rightarrow \ r \quad as \quad n \ \rightarrow \ 0 \quad .$$

It is evident that in this case λ is a *semistable measure* (see [3], for example), i. e. $\lambda \in ID(E)$ and $\lambda^r = a.\lambda * \delta(x)$ for some a and x.

A sequence (λ_n) is said to be *shift-convergent* if there exists a sequence (x_n) such that the sequence $(\lambda_n * \delta(x_n))$ weakly converges and to be compact if every its subsequence contains a convergent subsequence.

2. Main results

Theorem 1. *Let* $c > 0$, λ *be non-degenerated and* p *be a natural number. If there exist sequences* $(k(1,n)), ..., (k(p,n)), (a_n), (x_n)$ *and measures* $\mu_1, ..., \mu_p$ *such that*

$$a_n . \left(\mu_1^{*k(1,n)} \ * \ ... \ * \ \mu_p^{*k(p,n)}\right) \ * \ \delta\left(x_n\right) \quad \Rightarrow \quad \lambda \tag{2.1}$$

and $(1.2')$ *holds, then there exist sequences* $(t(1,n)), ..., (t(p,n)), (c_n)$, (y_n), *an element* y_0 *and measures* $\lambda_1, ..., \lambda_p \in ID(E)$ *such that* $t(i,n) \geq 1/c,\ n = 1,2,...;\ i = 1,...,p$ *and*

$$\lambda \quad = \quad \lambda_1 \ * \ ... \ * \ \lambda_p \quad * \quad \delta\left(y_0\right) \ , \tag{2.2}$$

$$\lambda \quad = \quad (c_1 ... c_n) \ . \ \left(\lambda_1^{s(1,n)} \ * \ ... \ * \ \lambda_p^{s(p,n)}\right) \ * \ \delta\left(y_n\right) \quad ,$$

with $s(i,n) = t(i,1)...t(i,n), n = 1,2,...; i = 1,...,p$.

Conversely, if (2.2) holds and

$$s(i,n) \ \rightarrow \ \infty \quad as \quad n \ \rightarrow \ \infty \ , \quad i = 1,...,p \ , \tag{2.3}$$

then (2.1) is true.

The above theorem partially solves the problem of characterizing the limit distributions of partial summation schemes. In the following theorem, the problem is concerned with a special case, when in (2.1) $\mu_1, ..., \mu_p$ belong to domains of semi-attraction of some semistable probability measures:

Theorem 2. *Let sequences* $(a_n), (x_n), (k(1,n)), ..., (k(p,n))$ *, a measure* λ *and a number* $c > 0$ *be given. Suppose that (2.1) holds and there exist positive numbers* $s(i) < 1$

and semistable measures ν_i, $i = 1, ..., p$ *, such that* μ_i *belongs to the domain of* $s(i)-$*semi-attraction of* ν *, i. e.*

$$b_i(n) \ . \ \mu_i^{*m(i,n)} \ * \ \delta\,(x_i(n)) \quad \Rightarrow \quad \nu_i \quad as \quad n \ \to \ \infty \tag{2.4}$$

for some $(b_i(n))$, $(x_i(n))$ *and*

$$m(i,n)/m(i,n+1) \ \to \ s(i) \quad as \quad n \ \to \ \infty \ . \tag{2.5}$$

Then there exist positive numbers b_i , $t(i) \ \in \ [s(i), 1]$, $i = 1, ..., p$ *and an element* x_0 *such that*

$$\lambda \quad = \quad b_1 \ . \ \nu_1^{t(1)} \ * \ ... \ * \ b_p \ .nu_p^{t(p)} \quad * \quad \delta\,(x_0) \ . \tag{2.6}$$

This theorem giving a condition for the limit measure λ in (2.1) to be a convolution of semistable measures has been proved on the real line by Chibisova (Theorem 2 [2]) with the additional condition $(1.2')$. The next theorem is devoted to investigate another condition for this.

Theorem 3. *Suppose that (2.1) holds and*

$$a_{n+1}/ \ a_n \quad \to \quad a \tag{2.7}$$

$$k(i,n+1)/k(i,n) \to t(i) \quad as \quad n \ \to \ \infty \ , \ i = 1, ..., p \ , \tag{2.8}$$

for some positive number a *and* $t(1), ..., t(p)$ *such that*

$$t(1) \ < \ t(2) \ < \ ... \ < \ t(p) \ . \tag{2.9}$$

Then there exist semistable measures λ_i , $i = 1, ..., p$ *, and* y_o *such that*

$$\lambda \quad = \quad \lambda_1 \ * \ ... \ * \ \lambda_p \quad * \quad \delta\,(y_0) \ . \tag{2.10}$$

Moreover, there exist sequences $(z_i(n))$, $i \ = \ 1, \ ... \ , p$ *such that*

$$a_n.\mu_i^{*k(i,n)} \ * \ \delta\,(z_i(n)) \quad \Rightarrow \quad \lambda_i \ , \tag{2.11}$$

i. e. μ_i *belongs to the domain of semi-attraction of* $\lambda_i, i = 1, ..., p$.

Theorem 4. *If (2.1) , (2.7) and (2.8) hold then there exists a natural number* q *such that* λ *is a convolution of* q *semistable measures.*

3. Lemmas and proofs

First we introduce a lemma which will play a crucial role in the following development.

Lemma 1. *Let μ be nondegenerated, $\lambda_n, n = 1, 2, ...,$ and (x_n), $(k(n))$, $(m(n))$ be given. Suppose that*

$$\lambda_n^{k(n)} * \delta\,(x_n) \Rightarrow \mu \tag{3.1}$$

and

$$m(n)/k(n) \to t \geq 0 \ . \tag{3.2}$$

Then $\mu \in ID(E)$ and there exists a sequence (y_n) such that

$$\lambda_n^{*m(n)} * \delta\,(y_n) \Rightarrow \mu^t.$$

Proof. From (3.1) we can see that if P is any finite-dimensional linear projector of the space E then

$$(P\lambda_n)^{*k(n)} * \delta\,(Px_n) \Rightarrow P\mu.$$

Thus, by the classical argument on finite-dimensional spaces, we infer that $P\mu$ is infinitely divisible, consequently $\mu \in ID(E)$ in view of Corollary 1 [8, p.320]. Now, from (3.2) we can choose a natural number N such that $m(n) < N.k(n)$ for all n. Then (3.1) yields

$$\lambda_n^{*Nk(n)} * \delta\,(Nx_n) \Rightarrow \mu^{*N}.$$

On the other hand,

$$\lambda_n^{*Nk(n)} = \lambda_n^{*m(n)} * \lambda_n^{*(Nk(n)-m(n))}.$$

Hence, by virtue of Theorem III.2.2 [6], there exists a sequence (z_n) such that the sequence

$$\left(\lambda_n^{*m(n)} * \delta\,(z_n)\right) \tag{3.3}$$

is compact. Meanwhile, for every $y^{,} \in E^{'}$, (3.1) and (3.2) imply

$$(y^{,}\lambda_n)^{*m(n)} * \delta\,(< y^{,}, (m(n)/k(n)).x_n >) \Rightarrow (y^{,}\mu)^t.$$

Then, by the Convergence of Type Theorem, if ν is any cluster point of the sequence (3.3) then

$$(y^{\cdot}\nu = (y^{\cdot}\nu)^t * \delta\left(x_{y^{\cdot}}\right)$$

for some real $x_{y^{\cdot}}$. Thus, it follows from Corollary 1 of Lemma 2 [7] the existence of $x(\nu) \in E$ such that $< y^{\cdot}, x(\nu) >= x_{y^{\cdot}}$ for all $y^{\cdot} \in E^{'}$ and $\nu = \mu^t * \delta(x(\nu))$. The set

$$\Gamma = \{x(\nu)\ :\ \nu \quad \textit{is a cluster point of}\ (3.3)\}$$

is a compact set provided the compactness of (3.3). Let ρ denote the Levy-Prokhorov metric on $P(E)$. For every n we define $z_n(0)$ by

$$\rho\left(\lambda_n^{*m(n)} * \delta\left(z_n\right), \mu^t * \delta\left(z_n(0)\right)\right) =$$

$$= \min_{x\in\Gamma} \rho\left(\lambda_n^{*m(n)} * \delta\left(z_n\right), \mu^t * \delta(x)\right)$$

Then, it is evident that

$$\rho\left(\lambda_n^{*m(n)} * \delta\left(z_n - z_n(0)\right), \mu^t\right) \to 0$$

which implies

$$\lambda_n^{*m(n)} * \delta\left(y_n\right) \Rightarrow \mu^t$$

with $y_n = z_n - z_n(0)$, i. e. the conclusion of the lemma is true.

Proof of Theorem 1. We invoke Theorem III.5.1 [6] and (2.1) to deduce that there exist sequences $(y_i(n))$, $i = 1, ..., p$, such that the sequences

$$\left(a_n.\mu_i^{*k(i,n)} * \delta\left(y_i(n)\right)\right), \quad i = 1, ..., p \tag{3.4}$$

are compact. Then from $(1.2')$ there are a subsequence (n') , numbers $t(i, 1)$ and measures $\lambda_i, \nu_i \in ID(E)$, $i = 1, ..., p$, such that $t(i, 1) \geq 1/c$ and

$$k(i, n' + 1)/k(i, n') \to t(i, 1) \tag{3.5}$$

$$a_{n'}.\mu_i^{*k(i,n')} * \delta\left(y_i(n')\right) \Rightarrow \lambda_i \tag{3.6}$$

$$a_{n'+1}.\mu_i^{*k(i,n'+1)} * \delta\left(y_i(n' + 1)\right) \Rightarrow \nu_i \tag{3.7}$$

for $n' \to \infty$ and $i = 1, ..., p$. Consequently, (2.1) yields

$$\lambda = \lambda_1 * ... * \lambda_p * \delta\left(y_0\right) \tag{3.8}$$

$$\lambda = \nu_1 * ... * \nu_p * \delta(z_0).$$

Moreover, in view of Theorem 1 [4] and Lemma 1, the conditions (3.5) through (3.7) imply

$$\nu_i = c_1.\lambda_i^{t(i,1)} * \delta(z_i)\ ,\ i = 1, ..., p,$$

for some positive c_1 and elements $z_i,\ i = 1, ..., p$. Hence

$$\lambda = c_1.\left(\lambda_1^{t(1,1)} * ... * \lambda_p^{t(p,1)}\right) * \delta(y_1),$$

with $y_1 = z_0 + z_1 + ... + z_p$.

Further, from the compactness of the sequence (3.4), we can pick from (n') another subsequence (n'') such that for some $t(i, 2) \geq 1/c$ and $\kappa_i \in ID(E),\ i = 1, ..., p$, we have

$$k(i, n'' + 2)/k(i, n'' + 1) \to t(i, 2),$$

$$a_{n''+2}.\mu_i^{*k(i,n''+2)} * \delta(y_i(n'' + 2)) \Rightarrow \kappa_i$$

as $n'' \to \infty$. Then repeating the above argument we can see that

$$\lambda = (c_1.c_2).\left(\lambda_1^{t(1,1).t(1,2)} * ... * \lambda_p^{t(p,1).t(p,2)}\right) * \delta(y_2),$$

with $c_2 > 0$ and $y_2 \in E$.

The continuation of the above process arrives at the conclusion that for each n

$$\lambda = (c_1...c_n).\left(\lambda_1^{s(1,n)} * ... * \lambda_p^{s(p,n)}\right) * \delta(y_n),$$

with $s(i, n) = t(i, 1)...t(i, n), t(i, j) \geq 1/c, c_j > 0, j = 1, ..., n;\ i = 1, ..., p$, which together with (3.8) implies (2.2).

Conversely, let (2.2) and (2.3) hold. Then it follows from Theorem III.5.1 [6] the existence of the sequences $(z_i(n)),\ i = 1, ..., p$, such that sequences $(\gamma_i(n)),\ i = 1, ..., p$, are compact, where

$$\gamma_i(n) = b_n.\lambda_i^{s(i,n)} * \delta(z_i(n)),\ n = 1, 2, ...$$

and $b_n = c_1...c_n$.

Let (n') be any subsequence of natural numbers. Since λ is nondegenerated and

$$\lambda = \gamma_1(n') * ... * \gamma_p(n') * \delta\left(y_{n'} - z_1(n') - ... - z_p(n')\right),$$

at least one of the subsequences $(\gamma_i(n')),\ i = 1, ..., p$, say $(\gamma_1(n'))$, has a nondegenerated clust point γ_1 , i. e.

$$b_{n''}.\lambda_1^{s(1,n''} * \delta\left(z_1(n'')\right) \Rightarrow \gamma_1 \quad as \quad n'' \to \infty$$

for some subsequence (n'') of (n'). Hence there exists an element $y' \in E^{'}$ such that $y'\gamma_1$ is nondegenerated and

$$b_{n''}.y'\lambda_1^{s(1,n''} * \delta\left(\langle y', z_1(n'')\rangle\right) \ \Rightarrow \ y'\gamma_1 \ .$$

Consequently, since $s(1, n'') \to \infty$ as $n'' \to \infty$, we see that $b_{n''} \to 0$ as $n'' \to \infty$. In conclusion, every subsequence $(b_{n'})$ of $(b_{n''})$ contains another subsequence tending to zero, this means $b_n \to 0$ as $n \to \infty$. Then, because for each index $i = 1, ..., p$ the set $\{(\lambda_i)^s :\ 0 \le s \le 1\}$ is compact (see Theorem 5[3]), it is plain that

$$b_n.\lambda_i^{s(i,n)-[s(i,n)]} \Rightarrow \delta(0) \quad as \quad n \to \infty,$$

where $[s]$ denotes the integer part of a number s. This implies by virtue of (2.2) that

$$b_n.\left(\lambda_1^{[s(1,n)]} * ... * \lambda_1^{[s(p,n)]}\right) * \delta\left(y_n\right) \ \Rightarrow \ \lambda,$$

which yields (2.1) with $\mu_i = \lambda_i$, $x_n = y_n,\ k_i(n) = [s(i, n)],\ i = 1, ..., p;\ n = 1, 2, ...$ The proof is just complete.

Proof of Theorem 2. By the same reason as in Theorem 1, there exist sequences $(y_i(n)),\ \ i = 1, ..., p$ such that the sequences (3.4) are compact. Then we can find a subsequence (n') of natural numbers and measure $\lambda_i \in ID(E),\ i = 1, ..., p$, such that

$$a_{n'}.\mu_i^{*k(i,n')} * \delta\left(y_i(n')\right) \ \Rightarrow \ \lambda_i \ , \ i = 1, ..., p \ . \tag{3.9}$$

Let's fix an index i. Then, without loss of generality, we can suppose that

$$m(i, n') \le k(i, n') < m(i, n' + 1).$$

Thus from (2.5) it is clear that the set

$$k(i, n')/m(i, n'+1) : n' \in (n')$$

is compact and we can suppose once more that

$$k(i, n')/m(i, n'+1) \to t(i) \; as \; n' \to \infty \tag{3.10}$$

for some $t(i) \in [s(i), 1]$. Therefore, taking Theorem 1 [4] and Lemma 1 into account, by virtue of (2.4), (3.9) and (3.10), we infer that $a_{n'}/b_i(n'+1) \to b_i$ and

$$\lambda_i = b_i.\nu_i^{t(i)} * \delta\left(y_i\right),$$

for some $b_i > 0,\; y_i \in E$. Hence the theorem is proved in view of (2.1) and (3.9). For the proof of Theorem 3 we need the following lemma:

Lemma 2. *Let $\mu \in ID(E),\; \lambda \in ID(E),\; (x_n),\; (t(n))$ and (a_n) be given. Suppose that*

$$a_n.\lambda^{t(n)} * \delta\left(x_n\right) \;\Rightarrow\; \mu \tag{3.11}$$

and

$$t(n)/t(n+1) \;\to\; t > 0\;. \tag{3.12}$$

Then there exist $a > 0$ and x such that

$$\mu^t = a.\mu * \delta(x)\;, \tag{3.13}$$

i. e. m is semistable.

Proof. From (3.11) and (3.12), by an argument analogous to that used for the proof of Lemma 1, we can infer that

$$a_n.\lambda^{t(n+1)} * \delta\left(y_n\right) \;\Rightarrow\; \mu^{1/t}$$

for some sequence (y_n). Meantime,

$$a_{n+1}.\lambda^{t(n+1)} * \delta\left(y_{n+1}\right) \;\Rightarrow\; \mu.$$

Thus, (3.13) follows from Theorem 1 [4] with $a = \lim_{n\to\infty}(a_{n+1}/a_n)$.

Proof of Theorem 3. It is evident that (1.2') is true in view of (2.9). Hence, by the same argument as in the proof of Theorem 1, we can see that

$$\lambda = \lambda_1 * ... * \lambda_p * \delta\left(y_0\right) \ , \tag{3.14}$$

$$\lambda = (a^n) \, . \left(\lambda_1^{t(1)^n} * ... * \lambda_p^{t(p)^n}\right) * \delta\left(y_n\right).$$

Then

$$a^{-n}.\lambda^{t(p)^{-n}} * \delta\left((a.t(p))^{-n}.y_n\right) = \lambda_1^{(t(1)/t(p))^n} * ... * \lambda_p^{(t(p-1)/t(p))^n} * \lambda_p.$$

Meanwhile, since for $i = 1, ..., p-1$ we have $t(i)/t(p) < 1$, Theorem 5 [3] yields

$$\lambda_i^{(t(i)/t(p))^n} \ \Rightarrow \ \delta(0).$$

Hence

$$a^{-n}.\lambda^{t(p)^{-n}} * \delta\left((a.t(p))^{-n}.y_n\right) \ \Rightarrow \ \lambda_p \ . \tag{3.15}$$

Therefore, it follows from Lemma 2 that λ_p is semistable. As we have seen in the proof of Theorem 1, there exists a sequence $(y_p(n))$ such that the sequence

$$\left(a_n.\mu_p^{*k(p,n)} * \delta\left(y_p(n)\right)\right) \tag{3.16}$$

is compact. Let ν_p be any clust point of the sequence (3.16). Then repeating the argument of the proof of Theorem 1 and the above part we can conclude that

$$a^{-n}.\lambda^{t(p)^{-n}} * \delta\left(x_p(n)\right) \ \Rightarrow \ \nu_p$$

for some sequence $(x_p(n))$. This together with (3.15) and Theorem 1 [4] implies the existence of an element $z(\nu_p) \in E$ such that

$$\nu_p = \lambda_p * \delta(z(\nu_p)).$$

Hence, by the same reason used in the proof of Lemma 1 and the compactness of the sequence (3.16) we obtain

$$a_n.\mu_p^{*k(p,n)} * \delta\left(z_p(n)\right) \ \Rightarrow \ \lambda_p \tag{3.17}$$

for some sequence $(z_p(n))$, i. e. (2.11) holds for $i = p$. Now, (2.1) and (3.17) imply the shift convergence of the sequence

$$\left(a_n.\left(\mu_1^{*k(1,n)} * ... * \mu_{p-1}^{*k(p-1,n)}\right)\right)$$

and by the same way as the above we get the semistability of λ_{p-1} and (2.11) for $i = p - 1$. The proof is complete after the p times repeated application of the above argument.

Proof of the Theorem 4. As in the proof of Theorem 3 we see that (3.14) hold. Then by a renumeration if necessary, we can suppose that

$$t(1) \leq t(2) \leq ... \leq t(p) \ .$$

Let's set

$$r(1) = t(j(1)), \nu_1 = \lambda_1 * ... * \lambda_{j(1)}$$

if $t(1) = ... = t(j(1)) < t(j(1) + 1)$,

$$r(2) = t(j(2)), \nu_2 = \lambda_{j(1)+1} * ... * \lambda_{j(2)}$$

if $t(j(1) + 1) = t(j(1) + 2) = ... = t(j(2)) < t(j(2) + 1)$,

$$.\ .\ .\ .\ .$$

$$r(q) = t(p), \nu_q = \lambda_{j(q-1)+1} * ... * \lambda_p$$

Then it follows straightly from (3.14) that

$$q \leq p, \quad r(1) < r(2) < ... < r(q)$$

and

$$\lambda = \nu_1 * ... * \nu_q * \delta\left(y_0\right),$$

$$\lambda = a^n.\left(\nu_1^{r(1)^n} * ... * \nu_q^{r(q)^n}\right) * \delta\left(y_n\right).$$

Hence, arguing as in the proof of Theorem 3, we get the conclusion of the theorem.

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Institute of Mathematics, 18 Hoang Quoc Viet, Cau Giay District,
Ha Noi, Viet Nam
E-mail address: Hodang54@Yahoo.com