A REMARKABLE STRUCTURE AND CONNECTIONS ON THE TANGENT BUNDLE

MONICA PURCARU AND MIRELA TĂRNAVEANU

Abstract. The present paper deals with the conformal almost symplectic structure on TM. Starting from the notion of conformal almost symplectic structure in the tangent bundle, we define the notion of general conformal almost symplectic d-linear connection and respective conformal almost symplectic d-linear connection with respect to a conformal almost symplectic structure $\hat{A}$, corresponding to the 1-forms $\omega$ and $\tilde{\omega}$ in $TM$. We determine the set of all general conformal almost symplectic d-linear connections on $TM$, in the case when the nonlinear connection is arbitrary and we find important particular cases.

1. Introduction

The geometry of the tangent bundle $(TM, \pi, M)$ has been studied by R. Miron and M. Anastasiei in [6], by R. Miron and M. Hashiguchi in [7], by V. Oproiu in [8], by Gh. Atanasiu and I. Ghinea in [1], by R. Bowman in [2], by K. Yano and S. Ishihara in [10], etc.

Concerning the terminology and notations, we use those from [4].

Let $M$ be a real $C^\infty$-differentiable manifold with dimension $n$, $(n=2n')$ and $(TM, \pi, M)$ its tangent bundle.

If $(x^i)$ is a local coordinates system on a domain $U$ of a chart on $M$, the induced system of coordinates on $\pi^{-1}(U)$ is $(x^i, y^i)$, $(i = 1, ..., n)$.

Let $N$ be a nonlinear connection on $TM$, with the coefficients $N^j_i(x, y), (i, j = 1, ..., n)$.

We consider on $TM$ an almost symplectic structure $A$ defined by:

$$A(x, y) = \frac{1}{2}a_{ij}(x, y)dx^i \wedge dx^j + \frac{1}{2}\tilde{a}_{ij}(x, y)\delta y^i \wedge \delta y^j, \quad (1)$$

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where \((dx^i, \delta y^i), (i = 1, ..., n)\) is the dual basis of \(\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)\), and \((a_{ij}(x, y), \tilde{a}_{ij}(x, y))\) is a pair of given \(d\)-tensor fields on \(TM\), of the type \((0,2)\), each of them alternate and nondegenerate.

We associate to the lift \(A\) the Obata’s operators:
\[
\left\{ \begin{array}{c}
\Phi^{ir}_{ij} = \frac{1}{2}(\delta^i_j \delta^r_j - a_{sj} a^{rj}), \\
\Phi^{sr}_{ij} = \frac{1}{2}(\delta^i_j \delta^r_j - a_{sj} a^{rj}), \\
\tilde{\Phi}^{ir}_{ij} = \frac{1}{2}(\delta^i_j \delta^r_j - \tilde{a}_{sj} \tilde{a}^{rj}), \\
\tilde{\Phi}^{sr}_{ij} = \frac{1}{2}(\delta^i_j \delta^r_j - \tilde{a}_{sj} \tilde{a}^{rj}).
\end{array} \right.
\] (2)

Obata’s operators have the same properties as the ones associated with a Finsler space [7].

Let \(\mathcal{A}_2(TM)\) be the set of all alternate \(d\)-tensor fields, of the type \((0,2)\) on \(TM\). As is easily shown, the relations on \(\mathcal{A}_2(TM)\) defined by (3):
\[
\left\{ \begin{array}{c}
(a_{ij} \sim b_{ij}) \iff ((3) \lambda(x, y) \in \mathcal{F}(TM), a_{ij}(x, y) = e^{2\lambda(x,y)} b_{ij}(x, y)), \\
(\tilde{a}_{ij} \sim \tilde{b}_{ij}) \iff ((3) \mu(x, y) \in \mathcal{F}(TM), \tilde{a}_{ij}(x, y) = e^{2\mu(x,y)} \tilde{b}_{ij}(x, y)),
\end{array} \right.
\] (3)
is an equivalence relation on \(\mathcal{A}_2(TM)\).

**Definition 1.1.** The equivalent class: \(\hat{A}\) of \(\mathcal{A}_2(TM)/\sim\) to which the almost symplectic tensor field \(A\) belongs, is called conformal almost symplectic structure on \(TM\).

Thus:
\[
\hat{A} = \{ A’ | A’_{ij}(x, y) = e^{2\lambda(x,y)} a_{ij}(x, y) \text{ and } \hat{A}_ij’(x, y) = e^{2\mu(x,y)} \tilde{a}_{ij}(x, y) \}. \] (4)

2. General conformal almost symplectic \(d\)-linear connections on \(TM\).

**Definition 2.1.** A \(d\)-linear connection, \(D\), on \(TM\), with local coefficients \(D\Gamma(N) = (L_i^j, \tilde{L}_i^j, \tilde{C}_i^j, C_i^j)\), is called general conformal almost symplectic \(d\)-linear connection on \(TM\) if:
\[
a_{ij} = K_{ijk}, \ a_{ij} | k = Q_{ijk}, \ \tilde{a}_{ij} | k = \tilde{K}_{ijk}, \ \tilde{a}_{ij} | k = \tilde{Q}_{ijk}, \] (5)

where \(K_{ijk}, Q_{ijk}, \tilde{K}_{ijk}, \tilde{Q}_{ijk}\) are arbitrary tensor fields, of the type \((0,3)\) on \(TM\), with the properties:
\[
K_{ijk} = -K_{jik}, \ Q_{ijk} = -Q_{jik}, \ \tilde{K}_{ijk} = -\tilde{K}_{jik}, \ \tilde{Q}_{ijk} = -\tilde{Q}_{jik}
\] (6)

and \(|\ \cdot \ |\) denote the \(h\)-and respective \(v\)-covariant derivatives with respect to \(D\).

Particularly, we have:

**Definition 2.2.** A \(d\)-linear connection, \(D\), on \(TM\), with local coefficients \(D\Gamma(N) = (L_i^j, \tilde{L}_i^j, \tilde{C}_i^j, C_i^j)\), for which there exists the \(1\)-forms \(\omega\) and \(\tilde{\omega}\) in \(TM\), \(\omega = \omega_i dx^i + \tilde{\omega}_i dy^i, \tilde{\omega} = \tilde{\omega}_i dx^i + \tilde{\omega}_i dy^i\) such that:
\[
\left\{ \begin{array}{c}
a_{ij} | k = 2\omega_k a_{ij}, \ a_{ij} | k = 2\tilde{\omega}_k a_{ij}, \\
\tilde{a}_{ij} | k = 2\tilde{\omega}_k \tilde{a}_{ij}, \ \tilde{a}_{ij} | k = 2\tilde{\omega}_k \tilde{a}_{ij},
\end{array} \right. \] (7)
where \( \mathbf{I} \) and \( \mathbf{I} \) denote the h-and v-covariant derivatives with respect to \( D \), is called conformal almost symplectic d-linear connection on \( TM \), with respect to the conformal almost symplectic structure \( \hat{A} \), corresponding to the 1-forms \( \omega, \tilde{\omega} \) and is denoted by: \( D\Gamma(N, \omega, \tilde{\omega}) \).

We shall determine the set of all general conformal almost symplectic d-linear connections, with respect to \( \hat{A} \).

Let \( \bar{D}\Gamma(N) = \left( L_{jk}^i, \bar{L}_{jk}^i, \bar{C}_{jk}^i, C_{jk}^i \right) \) be the local coefficients of a fixed d-linear connection \( \bar{D} \) on \( TM \). Then any d-linear connection, \( D \), on \( TM \), with local coefficients: \( D\Gamma(N) = \left( L_{jk}^i, \bar{L}_{jk}^i, \bar{C}_{jk}^i, C_{jk}^i \right) \), can be expressed in the form:

\[
\begin{align*}
N_{jk}^i &= N_{jk}^i - A_{jk}^i, \\
L_{jk}^i &= L_{jk}^i + A_k^i C_{jl}^i - B_{jk}^i, \\
\bar{L}_{jk}^i &= \bar{L}_{jk}^i + A_k^i \bar{C}_{jl}^i - \bar{B}_{jk}^i, \\
\bar{C}_{jk}^i &= \bar{C}_{jk}^i - \bar{D}_{jk}^i, \\
C_{jk}^i &= C_{jk}^i - D_{jk}^i, \\
A_{jk}^i &= 0,
\end{align*}
\]

where \( (A_{jk}^i, B_{jk}^i, \bar{B}_{jk}^i, \bar{D}_{jk}^i, D_{jk}^i) \) are components of the difference tensor fields of \( D\Gamma(N) \) from \( \bar{D}\Gamma(N) \), \([4]\) and \( 0, 0, 0 \) denotes the h-and v-covariant derivatives with respect to \( \bar{D} \).

**Theorem 2.1.** Let \( \bar{D} \) be a given d-linear connection on \( TM \), with local coefficients \( \bar{D}\Gamma(N) = \left( L_{jk}^i, \bar{L}_{jk}^i, \bar{C}_{jk}^i, C_{jk}^i \right) \). The set of all general conformal almost symplectic d-linear connections on \( TM \), with local coefficients \( D\Gamma(N) = \left( L_{jk}^i, \bar{L}_{jk}^i, \bar{C}_{jk}^i, C_{jk}^i \right) \) is given by:
Let $\tilde{D}$ be a given d-linear connection on $TM$, with local coefficients $\tilde{D}G(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, C^i_{jk})$. Then the following d-linear connection $D$, with local coefficients $DG(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, C^i_{jk})$ given by (10) is a general conformal almost symplectic d-linear connection with respect to $\tilde{A}$:

\[
\begin{align*}
N^i_j &= N^i_j - X^i_j, \\
L^i_{jk} &= L^i_{jk} + \frac{1}{2} \alpha^{is}(a^n_{sijk} - K_{sijk}), \\
\tilde{L}^i_{jk} &= \tilde{L}^i_{jk} + \frac{1}{2} \tilde{\alpha}^{is}(\tilde{a}^n_{sijk} - \tilde{K}_{sijk}), \\
\tilde{C}^i_{jk} &= \tilde{C}^i_{jk} + \frac{1}{2} \tilde{\alpha}^{is}(a^n_{sijk} - \tilde{K}_{sijk}), \\
C^i_{jk} &= C^i_{jk} + \frac{1}{2} \tilde{\alpha}^{is}(\tilde{a}^n_{sijk} - \tilde{K}_{sijk}),
\end{align*}
\]

where $X^i_j, X^i_{jk}, \tilde{X}^i_{jk}, \tilde{Y}^i_{jk}, Y^i_{jk}$ are arbitrary tensor fields on $TM$, $\tilde{1}, \tilde{1}$ denote the h-and respective v-covariant derivatives with respect to $\tilde{D}$ and $K_{ij}, Q_{ijk}, \tilde{K}_{ijk}, \tilde{Q}_{ijk}$ are arbitrary tensor fields of the type (0,3) on $TM$ with the properties (6).

**Particular cases:**

1. If $X^i_j = X^i_{jk} = \tilde{X}^i_{jk} = \tilde{Y}^i_{jk} = Y^i_{jk} = 0$ in Theorem 2.1, we have:

**Theorem 2.2.** Let $\tilde{D}$ be a given d-linear connection on $TM$, with local coefficients $\tilde{D}G(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, C^i_{jk})$. Then the following d-linear connection $D$, with local coefficients $DG(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, C^i_{jk})$ given by (10) is a general conformal almost symplectic d-linear connection with respect to $\tilde{A}$:

\[
\begin{align*}
L^i_{jk} &= L^i_{jk} + \frac{1}{2} \alpha^{is}(a^n_{sijk} - K_{sijk}), \\
\tilde{L}^i_{jk} &= \tilde{L}^i_{jk} + \frac{1}{2} \tilde{\alpha}^{is}(\tilde{a}^n_{sijk} - \tilde{K}_{sijk}), \\
\tilde{C}^i_{jk} &= \tilde{C}^i_{jk} + \frac{1}{2} \tilde{\alpha}^{is}(a^n_{sijk} - \tilde{K}_{sijk}), \\
C^i_{jk} &= C^i_{jk} + \frac{1}{2} \tilde{\alpha}^{is}(\tilde{a}^n_{sijk} - \tilde{K}_{sijk}),
\end{align*}
\]

where $\tilde{1}, \tilde{1}$ denote the h-and respective v-covariant derivatives with respect to the given d-linear connection $\tilde{D}$ and $K_{ij}, Q_{ijk}, \tilde{K}_{ijk}, \tilde{Q}_{ijk}$ are arbitrary tensor fields of the type (0,3) on $TM$ with the properties (6).

2. If $K_{ij} = \tilde{K}_{ijk} = \tilde{Q}_{ijk} = Q_{ijk} = 0$ in Theorem 2.1 we have :

**Theorem 2.3.** Let $\tilde{D}$ be a given d-linear connection on $TM$, with local coefficients $\tilde{D}G(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, C^i_{jk})$. The set of all almost symplectic d-linear connections on $TM$, with local coefficients $DG(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \tilde{C}^i_{jk}, C^i_{jk})$ is given by:
\[ \begin{align*}
N^i_j &= N^i_j - X^i_j, \\
L^i_{jk} &= L^i_{jk} + \bar{C}^i_{jm} X^m_k + \frac{1}{2} \alpha^i_s (a^o_s + a^0_s)_{m,k} - \delta^i_j \omega_k + \Phi^i h_j X^h_{rk}, \\
\bar{L}^i_{jk} &= \bar{L}^i_{jk} + \bar{C}^i_{jm} X^m_k + \frac{1}{2} \alpha^i_s (\bar{a}^o_s + \bar{a}^0_s)_{m,k} - \delta^i_j \bar{\omega}_k + \bar{\Phi}^i h_j \bar{X}^h_{rk}, \\
\bar{C}^i_{jk} &= \bar{C}^i_{jk} + \frac{1}{2} \alpha^i_s a^0_s_{k,j} - \delta^i_j \bar{\omega}_k + \bar{\Phi}^i h_j \bar{X}^h_{rk}, \\
C^i_{jk} &= C^i_{jk} + \frac{1}{2} \alpha^i_s \bar{a}^0_s_{k,j} - \delta^i_j \bar{\omega}_k + \bar{\Phi}^i h_j \bar{X}^h_{rk},
\end{align*} \]

where \( X^i_j, X^i_{jk}, \bar{X}^i_{jk}, \bar{Y}^i_{jk}, Y^i_{jk} \) are arbitrary tensor fields on \( TM \) and \( \bar{\omega}_i, \bar{\omega}_i, \bar{\omega}_i, \bar{\omega}_i, \bar{\omega}_i \) denote the \( h \)-and respective \( v \)-covariant derivatives with respect to \( \bar{D} \).

**Theorem 2.4.** Let \( \bar{D} \) be a given \( d \)-linear connection on \( TM \), with local coefficients \( \bar{\Gamma}^i(N) = (L^i_{jk}, \bar{L}^i_{jk}, \bar{C}^i_{jk}, C^i_{jk}) \). Then set of all conformal almost symplectic \( d \)-linear connections on \( TM \), with respect to \( \bar{A} \), corresponding to the 1-forms \( \omega \) and \( \bar{\omega} \), with local coefficients \( \bar{\Gamma}^i(N, \omega, \bar{\omega}) = (L^i_{jk}, \bar{L}^i_{jk}, \bar{C}^i_{jk}, C^i_{jk}) \) is given by:

\[ \begin{align*}
N^i_j &= N^i_j - X^i_j, \\
L^i_{jk} &= L^i_{jk} + \bar{C}^i_{jm} X^m_k + \frac{1}{2} \alpha^i_s (a^o_s + a^0_s)_{m,k} - \delta^i_j \omega_k + \Phi^i h_j X^h_{rk}, \\
\bar{L}^i_{jk} &= \bar{L}^i_{jk} + \bar{C}^i_{jm} X^m_k + \frac{1}{2} \alpha^i_s (\bar{a}^o_s + \bar{a}^0_s)_{m,k} - \delta^i_j \bar{\omega}_k + \bar{\Phi}^i h_j \bar{X}^h_{rk}, \\
\bar{C}^i_{jk} &= \bar{C}^i_{jk} + \frac{1}{2} \alpha^i_s a^0_s_{k,j} - \delta^i_j \bar{\omega}_k + \bar{\Phi}^i h_j \bar{X}^h_{rk}, \\
C^i_{jk} &= C^i_{jk} + \frac{1}{2} \alpha^i_s \bar{a}^0_s_{k,j} - \delta^i_j \bar{\omega}_k + \bar{\Phi}^i h_j \bar{X}^h_{rk},
\end{align*} \]

where \( X^i_j, X^i_{jk}, \bar{X}^i_{jk}, \bar{Y}^i_{jk}, Y^i_{jk} \) are arbitrary tensor fields on \( TM \), \( \omega = \omega_i dx^i + \bar{\omega}_i dy^i \) and respective \( \bar{\omega} = \bar{\omega}_i dx^i + \bar{\omega}_i dy^i \) are arbitrary 1-forms in \( TM \) and \( \bar{\omega}_i, \bar{\omega}_i, \bar{\omega}_i, \bar{\omega}_i, \bar{\omega}_i \) denote the \( h \)-and respective \( v \)-covariant derivatives with respect to \( \bar{D} \).
4. If \( X^i_j = \tilde{X}^i_j = \check{X}^i_j = Y^i_j = 0 \) in Theorem 2.4. we have:

**Theorem 2.5.** Let \( \bar{D} \) be a given \( d \)-linear connection on \( TM \), with local coefficients \( \bar{D} \Gamma(N) = (L^i_{jk}, \tilde{L}^i_{jk}, \check{C}^i_{jk}, C^i_{jk}) \). Then the following \( d \)-linear connection \( D \), with local coefficients \( D \Gamma(N, \omega, \tilde{\omega}) = (L^i_{jk}, \tilde{L}^i_{jk}, \check{C}^i_{jk}, C^i_{jk}) \) given by (13) is a conformal almost symplectic \( d \)-linear connection with respect to \( \hat{A} \), corresponding to the 1-forms \( \omega \) and \( \tilde{\omega} \):

\[
\begin{align*}
L^i_{jk} &= L^i_{jk} + \frac{1}{2} a^{is} a_{sjk} \omega_k - \delta^i_j \omega_k, \\
\tilde{L}^i_{jk} &= \tilde{L}^i_{jk} + \frac{1}{2} \tilde{a}^{is} \tilde{a}_{sjk} \omega_k - \delta^i_j \tilde{\omega}_k, \\
\check{C}^i_{jk} &= \check{C}^i_{jk} + \frac{1}{2} a^{is} a_{sjk} \tilde{\omega}_k - \delta^i_j \check{\omega}_k, \\
C^i_{jk} &= C^i_{jk} + \frac{1}{2} \tilde{a}^{is} \tilde{a}_{sjk} \check{\omega}_k - \delta^i_j \check{\omega}_k,
\end{align*}
\]

(13)

where \( \bar{L}, \tilde{L}, \check{C}, C \) denote the \( h \)-and respective \( v \)-covariant derivatives with respect to the given \( d \)-linear connection \( \bar{D} \) and \( \omega = \omega_i dx^i + \omega_i dy^i \) and respective \( \tilde{\omega} = \tilde{\omega}_i dx_i + \check{\omega}_i dy^i \) are two given 1-forms in \( TM \).

5. If we take an almost symplectic \( d \)-linear connection as \( \bar{D} \) in Theorem 2.5, then (13) becomes:

\[
\begin{align*}
L^i_{jk} &= L^i_{jk} - \delta^i_j \omega_k, \\
\tilde{L}^i_{jk} &= \tilde{L}^i_{jk} - \delta^i_j \tilde{\omega}_k, \\
\check{C}^i_{jk} &= \check{C}^i_{jk} - \delta^i_j \check{\omega}_k, \\
C^i_{jk} &= C^i_{jk} - \delta^i_j \check{\omega}_k,
\end{align*}
\]

(14)

6. If we take a conformal almost symplectic \( d \)-linear connection with respect to \( \hat{A} \) as \( \bar{D} \) in Theorem 2.4, we have

**Theorem 2.6.** Let \( \bar{D} \) be a given conformal almost symplectic \( d \)-linear connection on \( TM \), with local coefficients: \( \bar{D} \Gamma(N, \omega, \tilde{\omega}) = (L^i_{jk}, \tilde{L}^i_{jk}, \check{C}^i_{jk}, C^i_{jk}) \). The set of all conformal almost symplectic \( d \)-linear connections on \( TM \), with respect to \( \hat{A} \), corresponding to the 1-forms \( \omega \) and \( \tilde{\omega} \), with local coefficients \( D \Gamma(N, \omega, \tilde{\omega}) = (L^i_{jk}, \tilde{L}^i_{jk}, \check{C}^i_{jk}, C^i_{jk}) \) is given by:
\[
\begin{aligned}
&N^i_j = X^i_j - X^i_j, \\
&L^i_{jk} = L^i_{jk} + (C^i_{jm} + \delta^i_j \omega_m) X^m_k + \Phi^i_{rhj} X^h_r, \\
&\tilde{L}^i_{jk} = \tilde{L}^i_{jk} + (\tilde{C}^i_{jm} + \tilde{\delta}^i_j \tilde{\omega}_m) X^m_k + \tilde{\Phi}^i_{rhj} \tilde{X}^h_r, \\
&C^n_{jk} = C^n_{jk} + \Phi^n_{rhj} Y^h_r, \\
&\tilde{C}^n_{jk} = \tilde{C}^n_{jk} + \tilde{\Phi}^n_{rhj} \tilde{Y}^h_r, \\
&X^i_{jk} = 0,
\end{aligned}
\]

(15)

where \(X^i_j, X^i_{jk}, \tilde{X}^i_{jk}, Y^i_{jk}, \tilde{Y}^i_{jk}\) are arbitrary tensor fields on \(TM\), \(\omega = \omega_i dx^i + \dot{\omega}_i dy^i\) and respective \(\tilde{\omega} = \tilde{\omega}_i dx^i + \dot{\tilde{\omega}}_i dy^i\) are two arbitrary 1-forms in \(TM\) and \(D^0_i\) denote h-and respective v-covariant derivatives with respect to \(D^0\).

7. If we take \(X^i_j = 0\) in Theorem 2.6 we obtain:

**Theorem 2.7.** Let \(D^0\) be a given conformal almost symplectic d-linear connection on \(TM\), with local coefficients: \(D^0(N, \omega, \tilde{\omega}) = (L^i_{jk}, \tilde{L}^i_{jk}, C^n_{jk}, \tilde{C}^n_{jk})\). The set of all conformal almost symplectic d-linear connections on \(TM\), with respect to \(\tilde{\Lambda}\), which preserve the nonlinear connection \(N\), corresponding to the 1-forms \(\omega\) and \(\tilde{\omega}\), with local coefficients \(D^0(N, \omega, \tilde{\omega}) = (L^i_{jk}, \tilde{L}^i_{jk}, C^n_{jk}, \tilde{C}^n_{jk})\) is given by:

\[
\begin{aligned}
&L^i_{jk} = L^i_{jk} + \Phi^i_{rhj} X^h_r, \\
&\tilde{L}^i_{jk} = \tilde{L}^i_{jk} + \tilde{\Phi}^i_{rhj} \tilde{X}^h_r, \\
&C^n_{jk} = C^n_{jk} + \Phi^n_{rhj} Y^h_r, \\
&\tilde{C}^n_{jk} = \tilde{C}^n_{jk} + \tilde{\Phi}^n_{rhj} \tilde{Y}^h_r,
\end{aligned}
\]

(16)

where \(X^i_j, X^i_{jk}, \tilde{X}^i_{jk}, Y^i_{jk}, \tilde{Y}^i_{jk}\) are arbitrary tensor fields on \(TM\).

**References**


