ON THE CONVERGENCE OF THE SOLUTION OF THE QUASI-STATIC CONTACT PROBLEMS WITH FRICTION USING THE UZAWA TYPE ALGORITHM

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Dedicated to Professor Gheorghe Micula at his 60th anniversary

Abstract. The aim of the paper is to prove the convergence of a Uzawa type algorithm for a dual mixed variational formulation of a quasi-static contact problem with friction. This problem is considered as a saddle point problem which is approximated with the mixed finite element, where the stress, displacement and tangential displacement on the contact boundary will be simultaneously computed.

1. Introduction

The quasi-static model of the contact problems with friction, without the inertia effects, was proposed by [14] and consists of the formulation obtained through the approximation with finite differences of the variational inequality. The proof of the existence and uniqueness is based on the hypothesis that the displacements satisfy some conditions of regularity and the friction coefficient is small enough. The static contact problem with friction cannot describe the evolutive state of the contact conditions. For of this reason, the quasi-static formulation, of the contact problem with friction is preferred, which contains a dynamic formulation of the contact conditions and the inertial term is no longer used. Through the temporal discretization of the quasi-static contact problem, the so called incremental problem is obtained, equivalent with a sequence of static contact problems. Therefore, the quasi-static problem is solved step by step, at each time small deformations and displacements are calculated and are added at those calculated previously, as a result of a few small modifications of the applied forces, of the contact zone and of the contact conditions. Although, at each increment the dependence of the load-way is neglected, this hypothesis takes into account the way the applied forces change (modify themselves). From a mathematical point view, the problem obtained at each step is similar with a static problem.

This dual mixed variational formulation problem is descretized by the mixed finite element method and an Uzawa type algorithm is proposed. The iterative formulation of this algorithm is deduced and its convergence is proved.
The existence of solutions for the discrete problem by the mixed element method was obtained by Haslinger [7]. The contact problem has been recently studied by Andersen [11] and Rocca and Cocou [6] who proved that there exists a solution if the friction coefficient is small enough, and smooth and the contact functional is regular.

In this article is assumed that normal component of the stress vector and the contact zone is known.

2. Classical and variational formulation

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $3$, the polygonal domain occupied by a linear elastic body, and its boundary is denoted by $\Gamma$. Let $\Gamma_1, \Gamma_2$ and $\Gamma_c$ be three open disjoint parts of $\Gamma$ such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_c$, $\Gamma_1 \cap \Gamma_c = \emptyset$ and mes $(\Gamma_1) > 0$. We assume for the simplicity that $\Gamma_c$ is a segment for $d = 2$ and a polygon for $d = 3$. We denote by $u = (u_1, \ldots, u_d)$ the displacement field, $e = (\epsilon_{ij}(u)) = \left( \frac{1}{2} (u_{i,j} + u_{j,i}) \right)$ the strain tensor and $\sigma = (\sigma_{ij}(u)) = (a_{ijkl}\epsilon_{kl}(u))$ the stress tensor with the usual summation convention, where $i, j, k, l = 1, \ldots, d$. For the normal and tangential components of the displacement vector and stress vector, we use the following notation: $u_N = u_i - n_i$, $u_T = u - u_N \cdot n$, $\sigma_N = \sigma_{ij} u_i n_j$, $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_{N} n_i$, where $n = (n_i)$ is the outward unit normal vector to $\partial \Omega$.

Let us denote by $f$ and $h$ the density of body forces and traction forces, respectively. We assume that $a_{ijkl} \in L^\infty(\Omega)$, $l \leq i, j, k, l \leq d$, with usual condition of symmetry and elasticity, that is

$$a_{ijkl} = a_{ijlk} = a_{klji}, \quad l < i, j, k, l \leq d.$$

In this conditions, the fourth-order tensor $a = (a_{ijkl})$ is invertible a.e. on $\Omega$ and we denote its inverse $b = (b_{ijkl})$, and $\epsilon_{ij}(u) = (b_{ijkl}\sigma_{kl}(u))$, $i, j, k, l = 1, \ldots, d$.

The classical contact problem with dry friction in elasticity is which the normal stress $\sigma_N(u)$ and $\Gamma_c$ is assumed known, is follows: Find $u = u(x, t)$ such that $u(0, \cdot) = u^0(\cdot)$ in $\Omega$ and all $t \in [0, T]$, where $u^0$ is denoted the initial displacement of the body.

Condition (2.6) defines a form of Coulomb’s law of friction for elastostatic problems: $\mu_F$ is the coefficient of friction $\mu_F \in L^\infty(\Gamma_c)$, $\mu_F \geq \mu_0$ a.e. on $\Gamma_c$. 126
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The dual mixed variational formulation of the (2.1) - (2.6) in which stress, displacement and tangential displacement on contact zone are considerate unknown, it is shown the saddle-point problem with the form:

Find \((\sigma, u, \lambda) \in S_t \times V \times \Lambda\) for all \(t \in [0, T]\), such that

\[
L(\sigma, v, \mu) \leq L(\sigma, u, \lambda) \leq L(\tau, v, \mu) \quad \forall \ (\sigma, v, \mu) \in S_0 \times V \times \Lambda,
\]

where \(u \in W^{1,2}(0, T; V)\), \(\sigma \in W^{1,2}(0, T; S)\), \(f \in W^{1,2}(0, T; [L^2(\Omega)]^d)\), \(h \in W^{1,2}(0, T; [L^2(\Gamma)]^d)\) with \(\text{supp}(h(t)) \subset \Gamma_2\) for all \(t \in [0, T]\).

\[
L(\tau, v, \mu) = J_0(\tau) - (\text{div} \tau, \dot{v}) - < \tau, \mu >_{\Gamma_c}
\]

\[
J_0(\tau) = \frac{1}{2} a^*(\sigma, \tau) + (f, \text{div} \sigma + \dot{u})
\]

\[
t = \mu_F|\sigma_N(u)|, \quad \text{and} \quad \mu = |u_T| \text{ on } \Gamma_c
\]

\[
S_0 = \{\tau \mid \tau_{ij}, \tau_{ij,j} \in L^2(\Omega), \ \tau_{ij} = \tau_{ji}, \ \tau \cdot n = 0 \ a.e. \ on \ \Gamma^f_2\}
\]

\[
S_t = \{\tau \mid \tau_{ij}, \tau_{ij,j} \in L^2(\Omega), \ \tau_{ij} = \tau_{ji}, \ \tau \cdot n = t \ a.e. \ on \ \Gamma_2\}
\]

\[
S = \{\tau \mid \tau_{ij} \in L^2(\Omega), \ \tau_{ij} = \tau_{ji}, \ \tau_{ij,j} \in L^2(\Omega)\}
\]

endowed with inner product

\[
(\sigma, \tau)_S = \int_\Omega \sigma_{ij} \tau_{ij} \, dx.
\]

Norm \(\| \cdot \|_S\) is then

\[
\| \tau \|_S = (\tau, \tau)_S^{1/2}
\]

and

\[
a^*(\sigma, \tau) = \int_\Omega b_{ijkl} \sigma_{kl} \, dx.
\]

\(\Gamma^f_2\) can be regarded as part of \(\Gamma_2\) where \(h \equiv 0\),

\[
\Lambda = \{\mu \in H^{1/2}_0(\Gamma_c) \mid \mu \geq 0 \ on \ \Gamma^f_2\}
\]

\[
V = \{v \in H^1(\Omega) \mid v/\Gamma_1 = 0\}
\]

\[
H^{1/2}_0(\Gamma_c) = \{\mu \in H^{1/2}(\Gamma_c) \mid \rho^{-1/2} \mu \in L^2(\Gamma_c)\}
\]

The norm of \(H^{1/2}_0(\Gamma_c)\) is defined by

\[
\|\mu\|_{1/2, \Gamma_c} = \left\{\|\mu\|_{1/2, \Gamma_c}^2 + \|d^{-1/2} \mu\|_{0, \Gamma_c}^2\right\}^{1/2},
\]

where \(d\) denotes the distance between the point on \(\Gamma_c\) and the end point of \(\Gamma_c\) see [4].
3. The time discretisation and the mixed finite element approximation of the saddle point problem

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded and \((T_h)\) a triangulation of \( \Omega \). We assume that each triangulation is compatible with the partition of \( \Gamma \). i.e. each point where the boundary condition changes is a node of a set \( \Omega \), where \( \Omega = \cup_{i \in J_h} \Omega_i \), with \( \Omega_k \cup \Omega_l = \emptyset \) for all \( k, l \in J_h, k \neq l \).

The finite element approximation to the saddle-point problem (2.7) is as follow:

Find \((\sigma_h, u_h, \lambda_h) \in S^h \times V_h \times \Lambda_h \) for all \( t \in [0,T] \), such that:

\[
L(\sigma_h, u_h, \mu_h) \leq L(\sigma_h, u_h, \lambda_h), \forall (\tau_h, v_h, \mu_h) \in S^h \times V_h \times \Lambda_h \quad (3.1)
\]

where \( S^h = S_0 \cap S_h \), \( S^h = S_h \), \( \Lambda_h = M_h \cap \Lambda \) and \( S_h, V_h, M_h \) are subspaces of finite elements of \( S, V \) and \( H_{00}^{1/2}(\Gamma_c) \), respectively. Let \( S_h \) be \( RT_1 \), Raviart-Thomas space, \( V_h \) the space of the piecewise constant and \( M_h \) piecewise continuous linear subspace of \( H_{00}^{1/2}(\Gamma_c) \), is called the mortar space [10], as well.

We assume that the initial displacement field \( u \) satisfies the compatibility conditions, see ([8]).

The discrete Babuška-Brezzi condition should be satisfied for the dual mixed finite element method. It means to find an interpolation operator \( \pi_h \) from \( S \) to \( \Omega^h \), such that:

\[
b(\tau - \pi_h \tau, v_h, \lambda_h) = 0 \quad (3.2)
\]

\[
\|\pi_h \tau\| \leq c\|\tau\|, \forall \tau \in S,
\]

that means, for all \( \pi_h \tau \in S_h \) we have

\[
\int_{\Omega} \text{div}(\tau - \pi_h \tau) v_h dx + \int_{\Gamma_c} (\nabla \tau - \pi_h \nabla \lambda_h) \mu_h ds = 0, \quad (\forall v_h \in V_h, \mu_h \in \Lambda_h). \quad (3.4)
\]

Let

\[
\int_{\Omega} \text{div}(\tau - \pi_h \tau) v_h dx = 0, \quad (\forall v_h \in V_h) \quad (3.5)
\]

\[
\int_{\Gamma_c} (\nabla \tau - \pi_h \nabla \lambda) \mu_h ds = 0, \quad (\forall \mu_h \in \Lambda_h). \quad (3.6)
\]

Because \( \sigma_N(u) \) on \( \Gamma_c \) is regarded as given, applying Green’s formula to equation (3.5) in the finite element discrete form, is clear that the elements of subspace \( S_h \) satisfies (3.2) and (3.3) and we finally obtain further

\[
\|\tau_N h\|_{0,\Gamma_c} \leq \|\tau h\|_{0,\Omega} \leq \|\tau h\|_S, \quad (\forall \tau h \in S_h). \quad (3.7)
\]

The discretization of the saddle-point of the problem (3.1) by introduce a partition \( \{t_0, t_1, \ldots, t_N\} \) of time interval \([0, T]\) and consider on incremental formulation obtained by using the backward finite difference approximation of the time derivative of \( u \).

If we used \( u_h^k = u_h(x,t_h) \), \( \Delta u_h^k = u_h^{k+1} - u_h^k \), \( \Delta t = t^{k+1} - t^k \), \( \dot{u}_h(t^{k+1}) = \Delta u_h^k / \Delta t \), \( j_h^k = f_h(k\Delta t) \), \( h_h^k = h_h(k\Delta t) \), \( \sigma_h^k = \sigma_h(u_h^k) \), \( \lambda_h^k = \|u_h^k\|, \) for \( k = 0, 1, \ldots, N \)
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where \( \Delta t = \frac{T}{N} \)

In this case, we find \( (\sigma_h^k, u_h^k, \lambda_h^k) \in S^h \times V_h \times \Lambda_h \) such that

\[
L(\sigma_h^k, v_h^k, \mu_h^k) \leq L(\sigma_h^k, u_h^k, \lambda_h^k) \leq L(\tau_h^k, u_h^k, \lambda_h^k), \quad \forall (\tau_h^k, u_h^k, \mu_h^k) \in S^h \times V_h \times \Lambda_h, \quad (3.8)
\]

where \( \Lambda = \{\sigma_h^k, u_h^k, \lambda_h^k\} \) is the projector operator, \( \Lambda_h \) is a convex subset of \( H^{1/2}(\Gamma_c) \) to \( \Lambda_h \) is the convex subset of \( H_0^{1/2}(\Gamma_c), \rho > 0, s_h^k = \mu_F |\sigma_h^k|, k = 0, 1, \ldots, N. \)

4. Convergence analysis of the Uzawa algorithm

On the convergence (see [11]) with the finite element discrete problem (3.1) is following:

**Proposition 4.1.** If \( (\sigma_h^k, u_h^k, \lambda_h^k) \) is the saddle-point of the problem (3.8), then

(i) \[
J_0(\sigma_h^k, u_h^k) - (\text{div} \; \sigma_h^k, u_h^k) - < \mu_F |\sigma_h^k|, \lambda_h^k >_{\Gamma_c} \leq \]

(ii) \[
< \mu_F |\sigma_h^k|, \lambda_h^k >_{\Gamma_c} + (\text{div} \; \sigma_h^k + f^k, v_h^k - u_h^k) \leq 0,
\]

where \( \lambda_h^k = |v_h^k|, \mu_h^k = |u_h^k| \) on \( \Gamma_c, k = 0, 1, \ldots, N. \)

The proof can be deduced directly from the two inequalities showed at (3.8).

**Proposition 4.2.** The variational problem

\[
(\text{div} \; \sigma_h^k + f^k, v_h^k - u_h^k) + < \mu_F |\sigma_h^k|, \mu_h^k - \lambda_h^k >_{\Gamma_c} \leq 0 \quad (\forall \mu_h^k \in \Lambda_h, v_h^k \in V_h) \quad (4.1)
\]

is equivalent to

\[
\text{div} \sigma_h^k + f^k = 0, \quad \lambda_h^k = P_\Lambda(\rho s_h^k + \lambda_h^k) \quad (4.2)
\]

where \( P_\Lambda \) is the projection operator from \( L^2(\Gamma_c) \) to \( \Lambda_h \) is the convex subset of \( H^{1/2}(\Gamma_c), \rho > 0, \lambda_h^k = \mu_F |\sigma_h^k|, k = 0, 1, \ldots, N. \)

**Proof.** The ineqation (4.1) is equivalent to

\[
(\text{div} \sigma_h^k + f^k, u_h^k - v_h^k) + s_h^k, \lambda_h^k - \mu_h^k >_{\Gamma_c} \geq 0 \quad (\forall \mu_h^k \in \Lambda_h, v_h^k \in V_h). \quad (4.3)
\]

Multiplying the inequation (4.3) by \( \rho \) and adding \( (u_h^k - v_h^k, u_h^k) \) to the two sides of (4.3), we have

\[
(u_h^k - v_h^k, \rho(\text{div} \sigma_h^k + f^k) + u_h^k) + < \lambda_h^k - \mu_h^k, \rho s_h^k + \lambda_h^k >_{\Gamma_c} \geq
\]

\[
\geq (u_h^k - v_h^k, u_h^k) + < \lambda_h^k - \mu_h^k, \lambda_h^k >_{\Gamma_c}. \quad (4.4)
\]

But \( P_\Lambda \) is a projector operator,

\[
(u_h^k - v_h^k, \rho(\text{div} \sigma_h^k + f^k) + u_h^k) + (\lambda_h^k - \mu_h^k, P_\Lambda(\rho s_h^k + \lambda_h^k))_{\Gamma_c} \geq
\]

\[
\geq (u_h^k - v_h^k, u_h^k) + < \lambda_h^k - \mu_h^k, \lambda_h^k >_{\Gamma_c}. \quad (4.4)
\]
Hence
\[
(u_h^k - v_h^k, \rho(\nabla \sigma_h^k + f_h^k)) + (\lambda_h^k - \mu_h^k, P_\Lambda(\rho s_h^k + \lambda_h^k) - \lambda_h^k)_{0, \Gamma_c} \geq 0.
\] (4.5)

Because \(V_h\) and \(\Lambda_h\) are convex sets, we can put \((0 < \alpha < 1)\):
\[
\begin{aligned}
v_h^k &= (1 - \alpha)u_h^k + \alpha(\rho(\nabla \sigma_h^k + f_h^k) + u_h^k) \\
\mu_h^k &= (1 - \alpha)\lambda_h^k + \alpha P_\Lambda(\rho s_h^k + \lambda_h^k)
\end{aligned}
\] (4.6)

Substituting (4.6) in (4.5) yields
\[
\alpha(-\rho(\nabla \sigma_h^k + f_h^k), \rho(\nabla \sigma_h^k + f_h^k)) + \alpha(\lambda_h^k - P_\Lambda(\rho s_h^k + \lambda_h^k), P_\Lambda(\rho s_h^k + \lambda_h^k) - \lambda_h^k)_{0, \Gamma_c} \geq 0,
\]
that is equivalent with
\[
\alpha||\rho(\nabla \sigma_h^k + f_h^k)||_{0, \Omega}^2 + \alpha||\lambda_h^k - P_\Lambda(\rho s_h^k + \lambda_h^k)||_{0, \Gamma_c}^2 \leq 0 \quad (0 < \alpha < 1, \rho > 0),
\]
so we obtain
\[
\text{div} \sigma_h^k + f_h^k = 0 \quad \text{and} \quad \lambda_h^k = P_\Lambda(\rho s_h^k + \lambda_h^k), \rho > 0, \quad k = 0, 1, \ldots, N.
\]

From this results we can define the following Uzawa algorithm type:

a) Given \(u_h^{nk} \in V_h, \lambda_h^{nk} \in \Lambda_h\), we can define \(\sigma_h^{nk} \in S_h^1\) such that
\[
J_0(\sigma_h^{nk}) - (\nabla \sigma_h^{nk}, u_h^{nk}) - < s_h^{nk}, \lambda_h^{nk} >_{\Gamma_c} \leq 0
\]
\[
\leq J_0(\tau_h^{nk}) - (\nabla \tau_h^{nk}, u_h^{nk}) + < t_h^{nk}, \lambda_h^{nk} >_{\Gamma_c}, \quad \forall \tau_h^{nk} \in S_h^1.
\] (4.7)

b) Find \(u_h^{(n+1)k}\) and \(\lambda_h^{(n+1)k} = |v_h^{(n+1)k}|\) by using the following iterative method:
\[
u_h^{(n+1)k} = u_h^{nk} + \rho_n(\text{div} \sigma_h^{nk} + f_h^k)
\] (4.8)
\[
\lambda_h^{(n+1)k} = P_\Lambda(\rho_n s_h^{nk} + \lambda_h^{nk}),
\] (4.9)

when \(\rho_n > 0\) is chosen properly, \(k = 0, 1, \ldots, N.\)

We define the following bounded linear operator: \(g_r : S_h \to V \times L^2(\Gamma_c)\) by
\[
g_r(v, \mu) = (\nabla \tau, v) + < s, \mu >_{\Gamma_c}, \quad s = \mu(\sigma_N(v)), \quad \mu = |v|_{\tau}.
\]

Proposition 4.3. The operator \(g_r : S_h \to V \times L^2(\Gamma_c)\) is Lipschitz continuous, i.e. there exists a constant \(c > 0\), such that
\[
||g_r(\tau_1) - g_r(\tau_2)||_{V \times L^2(\Gamma_c)} \leq c||\tau_1 - \tau_2||_s, \quad \forall \tau_1, \tau_2 \in S_h,
\]
where \(\cdot ||\cdot_{V \times L^2(\Gamma_c)}\) denotes the norm of product space \(V \times L^2(\Gamma_c)\).

Proof is obtained from definition of \(g_r\) and from (3.7).

Theorem 4.4. There exists the constant \(\alpha_0\) and \(\alpha_1\), with \(0 < \alpha_0 \leq \rho_n \leq \alpha_1\), such that, the Uzawa type algorithm a)–b), is convergent in sense that \(\sigma_h^{nk} \to \sigma_h^k\) strongly in \(S\).


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Proof. We denote \( r_1^{n,k} = u_h^{n,k} - u_h^{k} \), \( r_2^{n,k} = \lambda_h^{n,k} - \lambda_h^{k} \), and from (4.7)-(4.9) we can deduce:

\[
\| r_1^{(n+1)k} \|_{0, \Omega}^2 + \| r_2^{(n+1)k} \|_{0, \Gamma_c}^2 = \| u_h^{(n+1)k} - u_h^k \|_{0, \Omega}^2 + \| \lambda_h^{(n+1)k} - \lambda_h^k \|_{0, \Gamma_c}^2 = \]

\[
= \| u_h^{n,k} + \rho_n (\text{div} \sigma_h^{n,k} + f^k) - u_h^{n,k} - \rho_n (\text{div} \sigma_h^{n,k} + f^k) \|_{0, \Omega}^2 + \nu (\rho_n \sigma_h^{n,k} + \lambda_h^{n,k}) - \nu (\rho_n \sigma_h^{n,k} + \lambda_h^{n,k}) \|_{0, \Gamma_c}^2 \leq \]

\[
\leq \| r_1^{n,k} \|_{0, \Omega}^2 + \| r_2^{n,k} \|_{0, \Gamma_c}^2 + 2\nu (r_1^{n,k}, \text{div}(\sigma_h^{n,k} - \sigma_h^k) + \nu (\sigma_h^{n,k} - \sigma_h^k) + \| (\sigma_h^{n,k} - \sigma_h^k) \|_{0, \Omega}^2 + \nu (\sigma_h^{n,k} - \sigma_h^k) \|_{0, \Gamma_c}^2 \]

\[
= \| r_1^{n,k} \|_{0, \Omega}^2 + \| r_2^{n,k} \|_{0, \Gamma_c}^2 + 2\nu (r_1^{n,k}, \text{div}(\sigma_h^{n,k} - \sigma_h^k)) + \nu (\sigma_h^{n,k} - \sigma_h^k) + \| (\sigma_h^{n,k} - \sigma_h^k) \|_{0, \Omega}^2 + \nu (\sigma_h^{n,k} - \sigma_h^k) \|_{0, \Gamma_c}^2 \]

With the Proposition 4.3 and (4.10) can be regarded as positive algebraic equations with degree two in \( \rho \), we get

\( a(\sigma_h^{n,k} - \sigma_h^k, \sigma_h^{n,k} - \sigma_h^k) + (r_1^{n,k}, \text{div}(\sigma_h^{n,k} - \sigma_h^k)) + < r_2^{n,k}, s_h^{n,k} - s_h^k > \leq 0 \),

where \( a \) is a linear symmetric form \( a : S \times S \to \mathbb{R} \), which with (4.10) implying:

\[
\| r_1^{(n+1)k} \|_{0, \Omega}^2 + \| r_2^{(n+1)k} \|_{0, \Gamma_c}^2 \leq \| r_1^{n,k} \|_{0, \Omega}^2 + \| r_2^{n,k} \|_{0, \Gamma_c}^2 - 2\nu (r_1^{n,k}, \text{div}(\sigma_h^{n,k} - \sigma_h^k)) + 2\nu (\sigma_h^{n,k} - \sigma_h^k) \leq 0 \]

\[
\| r_1^{n,k} \|_{0, \Omega}^2 + \| r_2^{n,k} \|_{0, \Gamma_c}^2 \leq 0 \]

For this inequation, we suppose \( 2\nu - 2\nu^2 \geq \beta > 0 \), and we choose \( \alpha_0 = \frac{1 - \sqrt{1 - 2\beta}}{2}, \alpha_1 = \frac{1 + \sqrt{1 - 2\beta}}{2} \) such that for \( \rho_n \in [\alpha_0, \alpha_1] \), then we have:

\[
\| r_1^{(n+1)k} \|_{0, \Omega}^2 + \| r_2^{(n+1)k} \|_{0, \Gamma_c}^2 + \beta \| \sigma_h^{n,k} - \sigma_h^k \|_S^2 \leq \| r_1^{n,k} \|_{0, \Omega}^2 + \| r_2^{n,k} \|_{0, \Gamma_c}^2 \]

(4.11)

From (4.11) results that the sequence \( (\| r_1^{n,k} \|_{0, \Omega}^2 + \| r_2^{n,k} \|_{0, \Gamma_c}^2 )_n \) is decreasing and has a finite limit, so that \( \beta \| \sigma_h^{n,k} - \sigma_h^k \|_S^2 \to 0 \) for \( n \to \infty \), and Theorem 4.4 is proved.

The solution \( \sigma_h^k \) of (3.8) is a fixed point of function \( M_h : S_h \to S_h \), so that \( \sigma_h^k \) is the limit of a sequence \( (\sigma_h^{n,k})_n \), defined by \( \sigma_h^{n,k} = M_h \sigma_h^{(n-1)k} \), (see [13]).

Theorem 4.5. In the conditions of Theorem 4.4, if \( \alpha_0 < \rho_n < \alpha_1 \) is true \( (\alpha_1 \) are chosen according to Theorem 4.4), then for the sequences \( \{ u_h^{n,k} \}_n, \{ \lambda_h^{n,k} \}_n \) defined by (4.8) - (4.9) we have:

1) \( \lim_{n \to \infty} \| u_h^{(n+1)k} - u_h^k \|_{0, \Omega} = 0, \lim_{n \to \infty} \| \lambda_h^{(n+1)k} - \lambda_h^k \|_{0, \Gamma_c} = 0 \);

2) \( \{ u_h^{n,k}, \lambda_h^{n,k} \}_n \to \{ u_h, \lambda_h \} \) weakly in \( V_h \times \Lambda_h \) where \( \{ u_h^k, \lambda_h^k \} \) is such that \( \sigma_h^k, u_h^k, \lambda_h^k \) is a saddle-point of \( L(\tau_h^k, v_h^k, \mu_h^k) \) on \( S_h^k \times V_h \times \Lambda_h \).
The proof is similar to that of Theorem 4.4, see [3].

5. Conclusions

We have analyzed, with Uzawa type algorithm of dual mixed variational formulation of the reduced version of a contact problem with friction in which it is assumed that the normal contact component of stress vector is known. For a more general contact problem, the existence solution is proved, but in very special cases.

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