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PRICING DIGITAL CALL OPTION IN THE HESTON STOCHASTIC VOLATILITY MODEL

VASILE L. LAZAR

Dedicated to Professor Gheorghe Micula at his 60^{th} anniversary

Abstract. The aim of this paper is to analyze the problem of digital option pricing under a stochastic volatility model, namely the Heston model (1993). In this model the variance v, follows the same square-root process as the one used by Cox, Ingersoll and Ross (1985) from the short term interest rate. We present an analytical solution for this kind of options, based on S. Heston's original work [3].

1. Introduction

Options on stock were first traded in an organized way on The Chicago Board Option Exchange in 1973, but the theory of option pricing has its origin in 1900 in "Théorie de la Spéculation" of L.Bachelier. In the early 1970's, after the introduction of geometric Brownian motion, Fischer Black and Myron Scholes made a major breakthrough by deriving the Black-Scholes formula which is one of the most significant results in pricing financial instruments [1].

We begin by presenting some underlying knowledge about basic concepts of derivatives and pricing methods.

A financial derivative is a financial instrument whose payoff is based on other elementary financial instruments, such as bonds or stocks. The most popular financial derivatives are: forward contracts, futures, swaps and options.

Options are particular derivatives characterized by non-negative payoffs. There are two basic types of option contracts: call options and put options.

Definition 1.1. *A* ***call*** *option gives the holder the right to buy a prescribed asset, the underlying asset, with a specific price, called the exercise price or strike price, at a specified time in future, called expiry or expiration date.*

Definition 1.2. *A* ***put*** *option gives the holder the right to sell the underlying asset, with an agreed amount at a specified time in future.*

The options can also be classified based on the time in which they can be exercised:

- A European option can only be exercised at expiry;

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- An American option can be exercised at any time up to and including the expiry

1.1. **Payoff Function.** Let S be the current price of the underlying asset and K be the strike price.Then, at expiry a European call option is worth:

$$max(S_T - K, 0) \tag{1.1}$$

This means that, the holder will exercise his right only if $S_T > K$ and than his gain is $S_T - K$. Otherwise, if $S_T \le K$, the holder will buy the underlying asset from the market and then the value of the option is zero.

The function (1.1) of the underlying asset is called **the payoff function**.

The payoff function from a European put option is:

$$max(K - S_T, 0) \tag{1.2}$$

Any option with a more complicated payoff structure than the usual put and call payoff structure is called an exotic option. In theory exists an unlimited number of possible exotic options but in practice there are only a few that have seen much use: digital or binary options, lookback options, barrier options, compound options, Asian options.

Digital options have a payoff that is discontinuous in the underlying asset price. For a digital call option with strike K at time T, the payoff is a Heaviside function:

$$DC(S\ ,\ T) = \mathcal{H}(S_T - K) = \begin{cases} 1 & \text{if } S_T \ge K \\ 0 & \text{if } S_T < K \end{cases} \tag{1.3}$$

and for a digital put option:

$$DP(S\ ,\ T) = \mathcal{H}(K - S_T) = \begin{cases} 1 & \text{if } S_T < K \\ 0 & \text{if } S_T \ge K \end{cases} \tag{1.4}$$

1.2. **Black-Scholes Formulae.** In 1973 Fischer Black and Myron Scholes derived a partial differential equation governing the price of an asset on which an option is based, and then solved it to obtain their formula for the price of the option, see [1].

We use the following notation:
S - the price of the underlying asset;
K - the exercise price;
t - current date;
T - the maturity date;
τ - time to maturity, $\tau = T - t$;
r - the risk free interest rate;
v - standard deviation of the underlying asset, i.e the volatility;
μ - the drift rate.

The assumptions used to derive the Black-Scholes partial differential equations are:

- the value of underlying asset is assumed to follow the log-normal distribution:

$$dS = \mu\ S\ dt + v\ S\ dW\ , \tag{1.5}$$

where the term $W(t)$ is a stochastic process with mean zero and variance t known as a Wiener process;
- the drift, μ, and the volatility, v, are constant throughout the option's life;
- there are no transaction costs or taxes;
- there are no dividends during the life of the option;
- no arbitrage opportunity;
- security trading is continuous;
- the risk-free rate of interest is constant during the life of the option.

Further, we give the most important result of stochastic calculus, Itô's lemma. Itô's lemma gives the rule for finding the differential of a function of one or more variables who follow a stochastic differential equation containing Wiener processes.

Lemma 1.1. *(One-dimensional Itô formula). Let the variable $x(t)$ follow the stochastic differential equation*

$$dx(t) \ = \ a(x \ , \ t) \ dt \ + \ b(x \ , \ t) \ dW.$$

Further, let $F(x(t) \ , \ t) \ \in \ C^{2,1}$ be at least a twice differentiable function. Then the differential of $F(x \ , \ t)$ is given by:

$$dF \ = \ \left[\frac{\partial F}{\partial x} a(x,t) \ + \ \frac{\partial F}{\partial t} \ + \ \frac{1}{2}\frac{\partial^2 F}{\partial x^2} b^2(x,t)\right] \ dt \ + \ \frac{\partial F}{\partial x} b(x,t) \ dW \ . \tag{1.6}$$

Proof: The proof of this lemma and the multi-dimensional case can be found in [4].

Using Itô's lemma and the foregoing assumptions, Black and Scholes have obtained the following partial differential equation for the option price $V(S \ , \ t)$:

$$\frac{\partial V}{\partial t} \ + \ \frac{1}{2} \ v^2 \ S^2 \ \frac{\partial^2 V}{\partial S^2} \ + \ r \ S \ \frac{\partial V}{\partial S} \ - \ r \ V \ = \ 0 \ . \tag{1.7}$$

In order to obtain a unique solution for the Black-Scholes equation we must consider final and boundary conditions. We will restrict our attention to a European call option, $C(S \ , \ t)$.

At maturity, $t \ = \ T$, a call option is worth:

$$C(S \ , \ T) \ = \ max(S_T \ - \ K \ , \ 0) \tag{1.8}$$

so this will be the final condition.

The asset price boundary conditions are applied at $S \ = \ 0$ and as $S \ \longrightarrow \ \infty$.

If $S \ = \ 0$ then dS is also zero and therefore S can never change. This implies on $S \ = \ 0$ we have:

$$C(0 \ , \ t) \ = \ 0 \ . \tag{1.9}$$

Obviously, if the asset price increases without bound $S \ \longrightarrow \ \infty$, then the option will be exercised indifferently how big is the exercise price. Thus as $S \ \longrightarrow \ \infty$ the value of the option becomes that of the asset:

$$C(S \ , \ t) \ \approx \ S \ , \ S \ \longrightarrow \ \infty \ . \tag{1.10}$$

We have now the following final-boundary value problem:

$$\begin{cases} \frac{\partial C}{\partial t} + \frac{1}{2} v^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C = 0 \\ C(0 , t) = 0 ; C(S , t) \approx S \text{ as } S \to \infty \\ C(S , T) = max(S_T - K , 0) \end{cases}$$

The analytical solution of this problem has the following functional form:

$$C(S , t) = S N(d_1) - K e^{-r(T-t)} N(d_2) \tag{1.11}$$

where

$$d_1 = \frac{log(S/K) + \left(r + \frac{1}{2} v^2\right) (T - t)}{v \sqrt{T - t}} \tag{1.12}$$

and

$$d_2 = \frac{log(S/K) + \left(r - \frac{1}{2} v^2\right) (T - t)}{v \sqrt{T - t}} \tag{1.13}$$

$N(x)$ is the *cumulative distribution function* for the standard normal distribution.

Similarly the price for a European put option is:

$$P(S , t) = - S N(- d_1) - K e^{-r(T-t)} N(- d_2) \tag{1.14}$$

In the digital option case, where we have the following final condition $DC(S , T) = \mathcal{H}(S_T - K)$, the solution for the option price equation is:

$$DC(S , t) = e^{-r(T-t)} N(d_2) \tag{1.15}$$

2. **Heston's Stochastic Volatility Model**

In the standard Black-Scholes model the volatility is assumed to be constant. Naturally the Black-Scholes assumption is incorrect and in reality volatility is not constant and it's not even predictable for timescales of more than a few months. This fact led to the development of stochastic volatility models, in which volatility itself is assumed to be a stochastic process.

We assume that S satisfies

$$dS = \mu S dt + v S dW_1 , \tag{2.1}$$

and, in addition the volatility follows the stochastic process:

$$dv = p(S , v , t) dt + q(S , v , t) dW_2 \tag{2.2}$$

where the two increments dW_1 and dW_2 have a correlation of ρ.

In this case the value V is not only a function of S and time t, it is also a function of the variance v, $V(S , v , t)$. The partial differential equation governing the option price is a generalization of Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} v^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho v S q \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + r S \frac{\partial V}{\partial S} + (p - \lambda q) \frac{\partial V}{\partial v} - r V = 0 . \tag{2.3}$$

where λ is the market price of volatility risk.

Examples of these models in continuous-time include Hull and White(1987), Johnson and Shanno(1987), Wiggins(1987), Stein and Stein(1991), Heston(1993), Bates(1996),and examples in discrete-time include Taylor(1986), Amin and Ng(1993) and Heston and Nandi(1993).

Among them, Heston's model is very popular because of its three main features:

- it does not allow negative volatility;
- it allows the correlation between asset return and volatility;
- it has a closed-form pricing formula.

Heston's option pricing formula is derived under the assumption that the stock price and its volatility follow the stochastic processes:

$$dS(t) \;=\; S(t)\;[\;\mu\;dt\;+\;\sqrt{v(t)}\;dW_1(t)\;] \tag{2.4}$$

and

$$dv(t) \;=\; k\;(\;\theta\;-\;v(t)\;)\;dt\;+\;\xi\;\sqrt{v(t)}\;dW_2(t)\;, \tag{2.5}$$

where:

$$\mathbf{Cov}[\;dW_1(t)\;,\;dW_2(t)\;]\;=\;\rho\;dt\;. \tag{2.6}$$

Finally, the market price of volatility risk is given by:

$$\lambda(S,v,t)\;=\;\lambda\;v\;. \tag{2.7}$$

According to the pricing equation (2.3) we have the following partial differential equation for the Heston model:

$$\begin{aligned}\frac{\partial V}{\partial t}\;+\;\frac{1}{2}\;v\;S^2\;\frac{\partial^2 V}{\partial S^2}\;+\;\rho\;\sigma\;v\;S\;\frac{\partial^2 V}{\partial S\;\partial v}\;+\;\frac{1}{2}\;\sigma^2\;v\;\frac{\partial^2 V}{\partial v^2}&\\ +\;r\;S\;\frac{\partial V}{\partial S}\;+\;[\;k\;(\theta\;-\;v\;)\;-\lambda\;v\;]\;\frac{\partial V}{\partial v}\;-\;r\;V\;=\;0\;.&\end{aligned} \tag{2.8}$$

The details of deriving the above equation and its closed-form solution, for a European call option, can be found in Heston's original work [3].

3. **A Closed-Form Solution for a Digital Call Option in the Heston Model**

In what follows we solve the partial differential equation (2.8) subject to the final condition:

$$DC(S\;,\;v\;,\;T)\;=\;\mathcal{H}(S_T\;-\;K)\;=\;\begin{cases}1 & \text{if } S_T\;\geq\;K\\ 0 & \text{if } S_T\;<\;K\end{cases} \tag{3.1}$$

In order to simplify our work it is convenient to make the following substitution $x\;=\;\ln[S]$, $U(x\;,\;v\;,\;t)\;=\;V(S\;,\;v\;,\;t)$. Then the equation (2.8) is turn into

$$\begin{aligned}\frac{\partial U}{\partial t}\;+\;\frac{1}{2}\;\sigma^2\;v\;\frac{\partial^2 U}{\partial v^2}\;+\;\rho\;\sigma\;v\;\frac{\partial^2 U}{\partial x\partial v}\;+\;\frac{1}{2}\;v\;\frac{\partial^2 U}{\partial x^2}\;+\;\left(r\;-\;\frac{1}{2}\;v\right)\;\frac{\partial U}{\partial x}&\\ +\;[k\;(\theta\;-\;v)\;-\;v\;\lambda]\;\frac{\partial U}{\partial v}\;-\;r\;U\;=\;0\;.&\end{aligned} \tag{3.2}$$

By analogy with the Black-Scholes formula (1.15), we guess a solution of the form:

$$DC(S\ ,\ v\ ,\ t)\ =\ e^{-r\ \tau}\ P \tag{3.3}$$

where the probability P correspond to $N(d_2)$ in the constant volatility case. P is the conditional probability that the option expires in-the-money:

$$P(x\ ,\ v\ ,\ T\ ;\ ln[K])\ =\ Pr[x(T)\ \geq\ ln[K]\ /\ x(t)\ =\ x\ ,\ v(t)\ =\ v]\ . \tag{3.4}$$

We now substitute the proposed value for $DC(S\ ,\ v\ ,\ t)$ into the pricing equation (3.2). We obtain:

$$\begin{aligned} &e^{-r\tau}\frac{\partial P}{\partial t}+rPe^{-r\tau}+\frac{1}{2}\sigma^2 ve^{-r\tau}\frac{\partial^2 P}{\partial v^2}+\rho\sigma ve^{-r\tau}\frac{\partial^2 P}{\partial x\partial v}+\frac{1}{2}ve^{-r\tau}\frac{\partial^2 P}{\partial x^2}+ \\ &+\left(r-\frac{1}{2}v\right)e^{-r\tau}\frac{\partial P}{\partial x}+[k(\theta-v)-v\lambda]e^{-r\tau}\frac{\partial P}{\partial v}-rPe^{-r\tau}=0\ . \end{aligned} \tag{3.5}$$

This implies that P must satisfy the equation:

$$\begin{aligned} &\frac{\partial P}{\partial t}\ +\ \frac{1}{2}\ \sigma^2\ v\ \frac{\partial^2 P}{\partial v^2}\ +\ \rho\ \sigma\ v\ \frac{\partial^2 P}{\partial x\partial v}\ +\ \frac{1}{2}\ v\ \frac{\partial^2 P}{\partial x^2}\ +\ \left(r\ -\ \frac{1}{2}\ v\right)\ \frac{\partial P}{\partial x} \\ &\qquad +\ [k\ (\theta\ -\ v)\ -\ v\ \lambda]\ \frac{\partial P}{\partial v}\ =\ 0 \end{aligned} \tag{3.6}$$

subject to the terminal condition:

$$P(x\ ,\ v\ ,\ T\ ;\ ln[K])\ =\ 1_{\{x\ \geq\ ln[K]\}}\ . \tag{3.7}$$

The probabilities are not immediately available in closed-form, but the next part shows that their characteristic function satisfy the same partial differential equation (3.6).

3.1. **The Characteristic Function.** Suppose that we have given the two processes

$$dx(t)\ =\ \left(r\ -\ \frac{1}{2}\ v(t)\right)\ dt\ +\ \sqrt{v(t)}\ dW_1(t) \tag{3.8}$$

$$dv(t)\ =\ [k\ (\theta\ -\ v(t))\ -\ \lambda\ v(t)]\ dt\ +\ \sigma\ \sqrt{v(t)}\ dW_2(t) \tag{3.9}$$

with

$$cov[dW_1(t)\ ,\ dW_2(t)]\ =\ \rho\ dt \tag{3.10}$$

and a twice-differentiable function

$$f(x(t)\ ,\ v(t)\ ,\ t)\ =\ E[g(x(T)\ ,\ v(T))\ /\ x(t)\ =\ x\ ,\ v(t)\ =\ v]\ . \tag{3.11}$$

From Itô's lemma we obtain:

$$\begin{aligned} df\ =\ &\left(\ \frac{1}{2}\ \sigma^2\ v\ \frac{\partial^2 f}{\partial v^2}\ +\ \rho\ \sigma\ v\ \frac{\partial^2 f}{\partial x\partial v}\ +\ \frac{1}{2}\ v\ \frac{\partial^2 f}{\partial x^2}\ +\ \left(r\ -\ \frac{1}{2}\ v\right)\ \frac{\partial f}{\partial x}\right. \\ &\left.+\ [k\ (\theta\ -\ v)\ -\ v\ \lambda]\ \frac{\partial f}{\partial v}\ +\ \frac{\partial f}{\partial t}\ \right)\ dt \\ &+\ \left(r\ -\ \frac{1}{2}\ v\right)\ \frac{\partial f}{\partial x}\ dW_1\ +\ [k\ (\theta\ -\ v)\ -\ v\ \lambda]\ dW_2 \end{aligned}$$

In addition, by iterated expectations, we know that $f(x(t)\ ,\ v(t)\ ,\ t)$ is a martingale, therefore the df coefficient must vanisch, i.e.,

$$\frac{1}{2}\,\sigma^2\,v\,\frac{\partial^2 f}{\partial v^2}\ +\ \rho\,\sigma\,v\,\frac{\partial^2 f}{\partial x \partial v}\ +\ \frac{1}{2}\,v\,\frac{\partial^2 f}{\partial x^2}\ +$$
$$\left(r\ -\ \frac{1}{2}\,v\right)\frac{\partial f}{\partial x}\ +\ [k\ (\theta\ -\ v)\ -\ v\ \lambda]\,\frac{\partial f}{\partial v}\ +\ \frac{\partial f}{\partial t}\ =\ 0\ . \tag{3.12}$$

Equation (3.11) imposes the final condition

$$f(x\ ,\ v\ ,\ T)\ =\ g(x\ ,\ v) \tag{3.13}$$

Depending on the choice of g, the function f represents different objects. Choosing $g(x\ ,\ v)\ =\ e^{i\varphi x}$ the solution is the characteristic function, which is available in closed form. In order to solve the partial differential equation (3.12) with the above condition we invert the time direction: $\tau\ =\ T\ -\ t$. This mean that we must solve the following equation:

$$\frac{1}{2}\,\sigma^2\,v\,\frac{\partial^2 f}{\partial v^2}\ +\ \rho\,\sigma\,v\,\frac{\partial^2 f}{\partial x \partial v}\ +\ \frac{1}{2}\,v\,\frac{\partial^2 f}{\partial x^2}\ +$$
$$\left(r\ -\ \frac{1}{2}\,v\right)\frac{\partial f}{\partial x}\ +\ [k\ (\theta\ -\ v)\ -\ v\ \lambda]\,\frac{\partial f}{\partial v}\ -\ \frac{\partial f}{\partial t}\ =\ 0 \tag{3.14}$$

subject to the initial condition:

$$f(x\ ,\ v\ ,\ 0)\ =\ e^{i\varphi x} \tag{3.15}$$

We guess a solution, from this equation, of the form:

$$f(x\ ,\ v\ ,\ \tau)\ =\ e^{C(\tau)\ +\ D(\tau)\ v\ +\ i\varphi x} \tag{3.16}$$

with initial condition $C(0)\ =\ D(0)\ =\ 0$.

This "guess" is due to the linearity of the coefficients.

Substituting the functional form (3.16) into equation (3.14) we find that:

$$\frac{1}{2}\,\sigma^2\,v\,D^2\,f\ +\ \rho\,\sigma\,v\,i\,\varphi\,D\,f\ -\ \frac{1}{2}\,v\,\varphi^2\,f\ +$$
$$\left(r\ -\ \frac{1}{2}\,v\right)\,i\,\varphi\,f\ +\ [k\ (\theta\ -\ v)\ -\ v\ \lambda]\,D\,f\ -\ (C'\ +\ D'\,v)\,f\ =\ 0$$

Therefore

$$v\ \left(\frac{1}{2}\,\sigma^2\,D^2\ +\ \rho\,\sigma\,i\,\varphi\,D\ -\ \frac{1}{2}\,\varphi^2\ -\ \frac{1}{2}\,i\,\varphi\ -\ (k\ +\ \lambda)\,D\ -\ D'\right)\ +$$
$$+\ (r\,i\,\varphi\ +\ k\,\theta\,D\ -\ C')\ =\ 0.$$

This can be reduce in two ordinary differential equations:

$$\text{a) } D'\ =\ \frac{1}{2}\,\sigma^2\,D^2\ +\ \rho\,\sigma\,i\,\varphi\,D\ -\ \frac{1}{2}\,\varphi^2\ -\ \frac{1}{2}\,i\,\varphi\ -\ (k\ +\ \lambda)\,D \tag{3.17}$$

and

$$\text{b) } C'\ =\ r\,i\,\varphi\ +\ k\,\theta\,D\ . \tag{3.18}$$

Basic theory on differential equation, including the Riccati equation, can be found in [7] a)We shall solve the Riccati differential equation

$$D' = \frac{1}{2}\,\sigma^2\,D^2 + (\rho\,\sigma\,i\,\varphi - k - \lambda)\,D - \frac{1}{2}\,\varphi^2 - \frac{1}{2}\,i\,\varphi$$

using the substitution:

$$D = -\frac{E'}{\frac{\sigma^2}{2}\,E}$$

It follows that

$$E" - (\rho\,\sigma\,i\,\varphi - k - \lambda)\,E' + \frac{\sigma^2}{2}\left(-\frac{1}{2}\,\varphi^2 - \frac{1}{2}\,i\,\varphi\right) = 0 \tag{3.19}$$

Then the characteristic equation is

$$x^2 - (\rho\,\sigma\,i\,\varphi - k - \lambda)\,x + \frac{\sigma^2}{4}\left(-\varphi^2 - i\,\varphi\right) = 0\,.$$

Consequently, if we make the following notation

$$d = \sqrt{(\rho\,\sigma\,i\,\varphi - k - \lambda)^2 - \sigma^2\,(-\varphi^2 - i\,\varphi\,)}\,,$$

then the equation (3.19) has the general solution

$$E(\tau) = A\,e^{x_1\tau} + B\,e^{x_2\tau}\,,$$

where

$$x_{1,2} = \frac{\rho\,\sigma\,i\,\varphi - k - \lambda \pm d}{2}\,.$$

The boundary conditions

$$\begin{cases} E(0) = A + B \\ A\,x_1 + B\,x_2 = 0 \end{cases}$$

yield

$$A = \frac{g\,E(0)}{g - 1}$$

$$B = -\frac{E(0)}{g - 1}$$

where $g = \frac{x_1}{x_2}$. Hence we obtain

$$E(\tau) = \frac{E(0)}{g - 1}\,(g\,e^{x_1\tau} - e^{x_2\tau})$$

$$E'(\tau) = \frac{E(0)}{g - 1}\,(g\,x_1\,e^{x_1\tau} - x_2\,e^{x_2\tau})$$

and thus

$$D(\tau) = -\frac{2}{\sigma^2}\,\frac{E'}{E} = -\frac{2}{\sigma^2}\,x_2\,\frac{e^{x_2\tau} - e^{x_1\tau}}{e^{x_2\tau} - g\,e^{x_1\tau}}$$

Therefore our equation has the following solution:

$$D(\tau) = \frac{k + \lambda + d - \rho\,\sigma\,\varphi\,i}{\sigma^2}\left[\frac{1 - e^{d\tau}}{1 - g\,e^{d\tau}}\right] \tag{3.20}$$

where

$$d = \sqrt{(\rho\,\sigma\,\varphi\,i - k - \lambda)^2 - \sigma^2\,(-\varphi^2 - i\,\varphi)} \tag{3.21}$$

$$g \; = \; \frac{\rho \, \sigma \, \varphi \, i \; - \; k \; - \; \lambda \; - \; d}{\rho \, \sigma \, \varphi \, i \; - \; k \; - \; \lambda \; + \; d} \tag{3.22}$$

b)The second equation can be solved by mere integration:

$$\begin{aligned} C(\tau) \; &= \; r \, i \, \varphi \, \tau \; + \; k \, \theta \int_{\tau}^{0} - \frac{E'(s)}{\frac{\sigma^2}{2} \, E(s)} \, ds \\ &= \; r \, i \, \varphi \, \tau \; - \; \frac{2 \, k \, \theta}{\sigma^2} \int_{\tau}^{0} \frac{E'(s)}{E(s)} \, ds \\ &= \; r \, i \, \varphi \, \tau \; - \; \frac{2 \, k \, \theta}{\sigma^2} \, \ln \frac{E(\tau)}{E(0)} \; . \end{aligned}$$

It follows that

$$C(\tau) = ri\varphi\tau + \frac{k\theta}{\sigma^2}\left[(k+\lambda+d-\rho\sigma\varphi i)\tau - 2\ln\left(\frac{1-ge^{d\tau}}{1-e^{d\tau}}\right)\right] \; . \tag{3.23}$$

3.2. **Solution of the Digital Call Option.** We can invert the characteristic functions to get the desired probabilities, using a standard result in probability,that is, if F(x) is a one-dimensional distribution function and f its corresponding characteristic function, then the cumulative distribution function $F(x)$ and its corresponding density function $\phi(x) \; = \; F'(x)$ can be retrieved via:

$$\phi(x) \; = \; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} \; f(t) \; dt \tag{3.24}$$

$$F(x) \; = \; \frac{1}{2} \; + \; \frac{1}{2\pi} \int_{0}^{\infty} \frac{e^{itx} \; f(-t) \; - \; e^{-itx} \; f(t)}{i \; t} \; dt \tag{3.25}$$

or

$$F(x) \; = \; \frac{1}{2} \; - \; \frac{1}{\pi} \int_{0}^{\infty} \mathcal{R}e \left[\frac{e^{itx} \; f(t)}{i \; t}\right] \; dt \tag{3.26}$$

This result is showed by J.Gil-Pelaez in [2].

Thus, we get the desired probability:

$$P(x \; , \; v \; , \; t \; ; \; \ln K) \; = \; \frac{1}{2} \; + \; \frac{1}{\pi} \int_{0}^{\infty} \mathcal{R}e \left[\frac{e^{-i\varphi \ln K} \; f(x \; , \; v \; , \; \tau \; , \; \varphi)}{i \; \varphi}\right] \; d\varphi \tag{3.27}$$

We can summarize the above relations in the following Theorem:

Theorem 3.1. *Consider a Digital call option in the Heston model, with a strike price of K and a time to maturity of τ. Then the current price is given by the following formula:*

$$DC(S \; , \; v \; , \; t) \; = \; e^{-r \; \tau} \; P$$

where the probability function, P is given by:

$$P(x \; , \; v \; , \; t \; ; \; \ln K) \; = \; \frac{1}{2} \; + \; \frac{1}{\pi} \int_{0}^{\infty} \mathcal{R}e \left[\frac{e^{-i\varphi \ln K} \; f(x \; , \; v \; , \; \tau \; , \; \varphi)}{i \; \varphi}\right] \; d\varphi$$

and the characteristic function is:

$$f(x \; , \; v \; , \; \tau) \; = \; e^{C(\tau) \; + \; D(\tau) \; v \; + \; i\varphi x}$$

where $C(\tau)$ and $D(\tau)$ are given by (3.23) and (3.20) respectively.

References

[1] Black, F & Scholes, M (1973) : *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81:637-654;
[2] Gil-Pelaez, L (1951): *Note on the Inversion Theorem*, Biometrica 38, 481-482;
[3] Heston, S (1993): *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, The Review of Financial Studies, 6:327-343;
[4] Korn, R & Korn, E (1999): *Option Pricing and Portfolio Optimization-Modern Methods of Financial Mathematics*, American Math.Soc.;
[5] Nagel, H. (1999): *Zur Optionsbewertung bei stochastischer Volatilität am Beispiel der DTB: Eine theoretische und empirische Analyse*, Inauguraldissertation zur Erlangung des Doktorgrades der Wirtschaftswissenschaftlichen Fakultät der Eberhard-Karls-Universität Tübingen;
[6] Precup, R (1997): *Ecuatii cu derivate partiale*, Transilvania Press;
[7] Rus, I.A. (1996): *Ecuatii diferentiale, ecuatii integrale si sisteme dinamice*, Transilvania Press;
[8] Shreve, S.E (1996): *Stochastic Calculus and Finance*, Lecture notes, Carnegie Mellon University;
[9] Wilmott, P (1999): *Derivatives: The Theory and Practice of Financial Engineering*, John Wiley & Sons Ltd;
[10] Wilmott, P & Dewynne, J & Howison, S (1993): *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press;
[11] Wystup, U et al. MathFinance, URL: *http://www.mathfinance.de*

Fraunhofer-Institut für Techno- und Wirtschaftsmathematik,
Gottlieb-Daimler-Str.,Geb. 49, 67663, Kaiserslautern, Germany
E-mail address: `vasilazar@yahoo.com`