SOME APPLICATIONS OF AN ASYMPTOTICAL FIXED POINT THEOREMS FOR INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

T. BARANYAI

Dedicated to Professor Gheorghe Micula at his 60th anniversary

Abstract. In this paper we will present an application of an asymptotical fixed point theorem for integral equation with deviating argument.

The following result is well known: ([1], [2])

**Theorem 1.** Let the following integral equation with deviating argument:

\[ x(t) = h(t) + \int_{a}^{t} f(s, x(g(s))) ds, \quad t \in [a, b]. \]  \( (1) \)

We suppose that:

(a) \( h \in C([a, b], [a, b]), \) \( h(a) = 0 \)

(b) \( g : [a, b] \rightarrow [a, b], \) \( a \leq g(t) \leq t \leq b \)

(c) \( f \in C([a, b] \times \mathbb{R}) \)

\( \exists L_f > 0, |f(t, u) - f(t, v)| \leq L_f |u - v| \) for all \( t \in [a, b], u, v \in \mathbb{R} \)

Then the equation (1) has an unique solution in \( C[a, b] \).

In proving of this theorem are apply the contraction principle for the following operator:

\[ A : C[a, b] \rightarrow C[a, b], \]

\[ A(x)(t) := h(t) + \int_{a}^{t} f(s, x(g(s))) ds, \quad t \in [a, b]. \]

In the following we prove the existence and the unicity of the solution of the integral equation (1) without using condition (b) for the operator \( g. \) In the proof of theorem 1 are use the Bielicki norm, but in the following theorem we use the Cebișev norm and an asymptotic fixed point principle.

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Let $X$ be a Banach space. We consider the following integral equation:

$$x(t) = h(t) + \int_a^t f(s, x(g(s)))ds, \quad t \in [a, b]$$

(2)

**Theorem 2.** We suppose that:

(a) $g \in C([a, b], [a, b])$
(b) $h \in C([a, b], [a, b])$, $h(a) = 0$
(c) $f \in C([a, b] \times X, X)$

$$\exists L_f > 0, \quad \|f(t, u) - f(t, v)\|_X \leq L_f \|u - v\|_X \quad \text{for all } t \in [a, b], \ u, v \in X.$$

Then the equation (2) has an unique solution in $C([a, b], X)$.

**Proof.** We consider the operator

$$A : C([a, b], X) \longrightarrow C([a, b], X)$$

$$A(x)(t) := h(t) + \int_a^t f(s, x(g(s)))ds,$$

Then the iterates of $A$ are:

$$A^2(x)(t) = h(t) + \int_a^t f(s, A(x)(g(s)))ds,$$

$$\ldots$$

$$A^{n+1}(x)(t) = h(t) + \int_a^t f(s, A^n(x)(g(s)))ds$$

We have the following estimations ([3]):

$$|A(x)(t) - A(y)(t)| \leq L_f \int_a^t |x(g(s)) - y(g(s))|ds \leq L_f \|x - y\|_C \frac{t - a}{1!}, \quad \forall t \in [a, b] \quad (\|\cdot\|_C \text{ is the Cesàro norm})$$

$$\leq L_f \|x - y\|_C \frac{t - a}{1!}, \quad \forall t \in [a, b]$$

$$|A^2(x)(t) - A^2(y)(t)| \leq L_f \int_a^t |x(A(s)) - y(A(s))|ds \leq$$

$$\leq L_f \|x - y\|_C \int_a^t \frac{s - a}{1!}ds \leq L_f^2 \|x - y\|_C \frac{(t - a)^2}{2!}, \quad \forall t \in [a, b]$$

$$\ldots$$

$$|A^k(x)(t) - A^k(y)(t)| \leq L_f^k \|x - y\|_C \frac{(t - a)^k}{k!}, \quad \forall t \in [a, b], \forall k \in \mathbb{N}$$

$$\|A^k(x) - A^k(y)\| \leq \frac{[L_f(b - a)]^k}{k!} \|x - y\|_C, \quad \forall k \in \mathbb{N}.$$
So there exists a natural number $k$ such that:

$$A^k \text{ is contraction with the contraction constant } \alpha = \frac{(L_f(b-a))^k}{k!} < 1.$$ 

Now we apply an asymptotical variant of contraction principle ([2]) and we have that, the integral equation (2) has an unique solution. Q.E.D.

Remarks.
1. When we take $X = \mathbb{R}^m$ we have a result for the following system of integral equations:

$$x_1(t) = h_1(t) + \int_a^t f_1(s, x_1(g(s)), \ldots, x_m(g(s)))ds$$

$$x_2(t) = h_2(t) + \int_a^t f_2(s, x_1(g(s)), \ldots, x_m(g(s)))ds \quad t \in [a, b]$$

$$\ldots$$

$$x_m(t) = h_m(t) + \int_a^t f_m(s, x_1(g(s)), \ldots, x_m(g(s)))ds$$

2. When $X = l^2(\mathbb{R})$ we have a result for the following infinit sistem of integral equations:

$$x_1(t) = h_1(t) + \int_a^t f_1(s, x_1(g(s)), \ldots, x_m(g(s)))ds, \quad t \in [a, b], \quad i \in \mathbb{N}^*.$$

References