SOME APPLICATIONS OF WEAKLY PICARD OPERATORS

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Dedicated to Professor Wolfgang W. Breckner at his 60th anniversary

Abstract. In this paper we give some applications of weakly Picard operators theory to linear positive approximation operators, to difference equations with deviating argument and to functional-integral equations.

1. Introduction

Let \((X, d)\) be a metric space and \(A : X \to X\) an operator. In this paper we shall use the following notations:

\[ F_A := \{ x \in X | A(x) = x \}; \]
\[ I(A) := \{ Y \subset X | A(Y) \subset Y, Y \neq \emptyset \}; \]
\[ A^0 := 1_X, A^1 := A, \ldots, A^{n+1} := A \circ A^n, \quad n \in \mathbb{N}. \]

By definition an operator \(A\) is weakly Picard operator (WPO) if the sequence of successive approximations, \((A^n(x))_{n \in \mathbb{N}}\) converges for all \(x \in X\) and the limit is a fixed point of \(A\). If the operator \(A\) is WPO and \(F_A = \{ x^* \}\), then by definition the operator \(A\) is Picard operator (PO). For an WPO \(A\) we consider the operator \(A^\infty\) defined by

\[ A^\infty : X \to X, \quad A^\infty(x) := \lim_{n \to \infty} A^n(x). \]

We have the following characterization of the WPOs.

**Theorem 1.1** (I. A. Rus [6], [7], [12]). An operator \(A\) is WPO if and only if there exists a partition of \(X\), \(X = \bigcup_{\lambda \in \Lambda} X_\lambda\), such that

\(a)\) \(X_\lambda \in I(A), \quad \forall \lambda \in \Lambda;\)

\(b)\) \(X_\lambda \in I(A), \quad \forall \lambda \in \Lambda;\)

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(b) \( A|_{X_\lambda} : X_\lambda \to X_\lambda \) is PO, \( \forall \, \lambda \in \Lambda \).

The aim of this paper is to give some applications of this theorem.

2. **Iterates of two variables Bernstein operator**

Let \( \overline{D} = \{(x, y) \in \mathbb{R}^2 \mid x, y \in R_+, \ x + y \leq 1\} \) and \( e_{ij} : \overline{D} \to R_+ \) be defined by \( e_{ij} := x^i y^j, \ i, j \in N \).

Let we denote by \( \| \cdot \|_C \) the Chebyshev norm on \( C(\overline{D}) \).

In what follow we consider the two variables Bernstein operator (see D. D. Stancu [13])

\[
B_n : C(\overline{D}) \to C(\overline{D}), \quad n \in N^*
\]

defined by

\[
B_n(f)(x, y) := \sum_{0 \leq i + j \leq n} \frac{n!}{i!j!(n-i-j)!} x^i y^j (1 - x - y)^{n-i-j} f\left( \frac{i}{n}, \frac{j}{n} \right). \tag{2.1}
\]

It is well known that ([13]):

\[
e_{00}, e_{01}, e_{10} \in F_{B_n}, \quad n \in N^*.
\]

We have

**Theorem 2.1.** The operator \( B_n \) is WPO and

\[
B_n^\infty (f)(x, y) = f(0, 0) + [f(1, 0) - f(0, 0)]x + [f(0, 1) - f(0, 0)]y, \quad x, y \in \overline{D}; \ f \in C(\overline{D}).
\]

**Proof.** Let

\[
X_{\alpha, \beta, \gamma} := \{f \in C(\overline{D}) \mid f(0, 0) = \alpha, \ f(1, 0) = \beta, \ f(0, 1) = \gamma\},
\]

\[
f_{\alpha, \beta, \gamma}(x, y) := \alpha + (\beta - \alpha)x + (\gamma - \alpha)y, \quad x, y \in \overline{D},
\]

for all \( \alpha, \beta, \gamma \in R \).

We remark that

(i) \( X_{\alpha, \beta, \gamma} \) is a closed subset of \( C(\overline{D}) \);

(ii) \( X_{\alpha, \beta, \gamma} \) is an invariant subset of \( B_n \), for all \( \alpha, \beta, \gamma \in R \) and \( n \in N^* \);

(iii) \( C(\overline{D}) = \bigcup_{\alpha, \beta, \gamma \in R} X_{\alpha, \beta, \gamma} \) is a partition of \( C(\overline{D}) \);

(iv) \( f_{\alpha, \beta, \gamma} \in X_{\alpha, \beta, \gamma} \cap F_{B_n} \).
Now we prove that
\[ B_n|_{X_{\alpha,\beta,\gamma}} : X_{\alpha,\beta,\gamma} \to X_{\alpha,\beta,\gamma} \]
is a contraction for all \( \alpha, \beta, \gamma \in R \) and \( n \in N^* \).

Let \( f, g \in X_{\alpha,\beta,\gamma} \). From (2.1) we have
\[
|B_n(f)(x, y) - B_n(g)(x, y)| = |B_n(f - g)(x, y)| \leq |1 - (1 - x - y)^n - x^n - y^n| \cdot \|f - g\|_C \leq \left(1 - \frac{1}{2^n-1}\right) \|f - g\|_C, \quad \forall \, x, y \in D.
\]
So,
\[
\|B_n(f) - B_n(g)\|_C \leq \left(1 - \frac{1}{2^n-1}\right) \|f - g\|_C, \quad \forall \, f, g \in X_{\alpha,\beta,\gamma};
\]
i.e., \( B_n|_{X_{\alpha,\beta,\gamma}} \) is a contraction for all \( \alpha, \beta, \gamma \in R \).

From the contraction principle \( f_{\alpha,\beta,\gamma} \) is the unique fixed point of \( B_n \) in \( X_{\alpha,\beta,\gamma} \) and that \( B_n|_{X_{\alpha,\beta,\gamma}} \) is a PO.

From the Theorem 1.1 the proof follows.

**Remark 2.1.** For the one dimensional case see I. A. Rus [10], [11], [12] and O. Agratini and I. A. Rus [1]. See also R.P. Kelisky and T.J. Rivlin [4].

**Remark 2.2.** The case \( D = [0, 1] \times [0, 1] \) (see P. L. Butzer [3]) will be presented elsewhere.

**Remark 2.3.** A similar result for Bernstein operators on a simplex we have.

### 3. Difference equations in \( C([0, 1], X) \)

Let \( X \) be a Banach space. We denote by \( \| \cdot \|_C \) the Chebyshev norm on \( C([0, 1], X) \). Let \( h \in C([0, 1] \times X \times X) \) and \( g \in C([0, 1] \times X, X) \) be two operators. In what follow we consider the following difference equation with deviating argument, in \( C([0, 1], X) \),
\[
x_{n+1}(t) = h(t, x_n(t), x_n(0)) + g(t, x_n(t)), \quad t \in [0, 1], \quad n \in N^* \tag{3.1}
\]
For to study this equation we consider the operator
\[
A : C([0, 1], X) \to C([0, 1], X)
\]
\[
A(x)(t) := h(t, x(t), x(0)) + g(t, x(t)).
\]
We have

**Theorem 3.1.** We suppose that

(i) \( h(0, \lambda, \lambda) = \lambda, \forall \lambda \in X \);
(ii) \( g(0, \lambda) = 0, \forall \lambda \in X \);
(iii) \( g(t, \cdot, \lambda) \) is an \( \alpha \)-contraction for all \( t \in [0, 1] \);
(iv) \( h(t, \cdot, \lambda) \) is a \( \beta \)-contraction for all \( t \in [0, 1], \lambda \in X \);
(v) \( \alpha + \beta < 1 \).

Then the operator \( A \) is WPO.

**Proof.** Let

\[ X_\lambda := \{ x \in C([0,1], X) | x(0) = \lambda \}, \lambda \in X. \]

Then

(a) \( X_\lambda \) is a closed subset of \( C([0,1], X) \);
(b) \( X_\lambda \in I(A) \), for all \( \lambda \in X \);
(c) \( C([0,1], X) = \bigcup_{\lambda \in A} X_\lambda \) is a partition of \( C([0,1], X) \).

From (i)-(v) we have that the restriction of \( A \) to \( X_\lambda \) is an \( (\alpha + \beta) \)-contraction.

By the Theorem 1.1 we have that the operator \( A \) is WPO.

Let \( x^{\ast}_\lambda \) be the unique fixed point of the operator \( A \) in \( X_\lambda \). It is clear that \( \text{card} F_A = \text{card} X \), and that \( F_A \) is the equilibrium solution set of the equation (3.1).

From the Theorem 3.1 we have

**Theorem 3.2.** In the conditions of the Theorem 3.1, let \( (x_n)_{n \in N} \) be a solution of the equation (3.1). If \( x_0 \in X_\lambda \), then \( x_n \in X_\lambda \), for all \( n \in N \). Moreover

\[ x_n \to x^{\ast}_\lambda \text{ as } n \to \infty. \]

**Remark 3.1.** In the conditions of Theorem 3.1 the equilibrium solution \( x^{\ast}_\lambda \) is globally asymptotically stable relative to \( X_\lambda \).

**Remark 3.2.** For the fixed point technique in the theory of difference equations see M. A. Şerban [14].

**Remark 3.3.** The following example is in the conditions of the Theorem 3.1:

\[ x_{n+1}(t) = \frac{1}{2} t \sin x_n(t) + x_n(0), \quad n \in N \]
$x_0 \in C[0,1]$

4. Functional-integral equations

Let $X$ be a Banach space $f \in C([a,b] \times X, X)$ and $K \in C([a,b] \times [a,b] \times X, X)$. Consider the following functional-integral equation

$$x(t) = x(a) + \int_a^t f(s, x(s))ds + \int_a^t \int_a^s K(s, u, x(u))duds \quad (4.2)$$

$t \in [a, b]; \quad x \in C([a, b], X)$$

Let $X_\lambda := \{ x \in C([a, b], X) \mid x(a) = \lambda \}, \lambda \in X$ and $A : C([a, b], X) \to C([a, b], X)$ defined by $A(x)(t) := \text{second part of (4.1)}$.

If we denote by $S$ the solution set of the eq. (4.1) then $S = F_A$.

We remark that

(a) $X_\lambda$ is a closed subset of $C([0,1], X)$ for all $\lambda \in X$;
(b) $X_\lambda \in I(A)$;
(c) $C([0,1], X) = \bigcup_{\lambda \in X} X_\lambda$ is partition of $C([0,1], X)$;
(d) if $f(s, \cdot)$ is $L_f$-Lipschitz and $K(s, u, \cdot)$ is $L_K$-Lipschitz for all $s, u \in [a,b]$ then the restriction of $A$ to $X_\lambda$ is a contraction with respect to a suitable Bielecki’s norm. More exactly if we denote

$$\|x\|_B = \max_{a \leq t \leq b} (\|x(t)\|e^{-r(t-a)})$$

then we have

$$\|A(x) - A(y)\|_B \leq \left( \frac{L_f}{r} + \frac{L_K}{r^2} \right) \|x - y\|_B, \forall x, y \in X_\lambda; \lambda \in X.$$

Let $x_\lambda^*$ be the unique fixed point of $A$ in $X_\lambda$. From the Theorem 1.1 it follows that the operator $A$ is WPO and $\text{card}F_A = \text{card}X$.

So, we have

**Theorem 4.1.** In the above conditions

(1) $\text{card}S = \text{card}X$

(2) the solution $x_\lambda^*$ is globally asimptotically stable with respect to $X_\lambda$.

**Remark 4.1.** For other types of functional integral equations see R. Precup [5], I. A. Rus [8] and [9].
Remark 4.2. For other applications of the WPO see A. Buică [2], I. A. Rus [6], [7].

References


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