A NOTE ON THE DIVISIBILITY OF SOME COMPRESSION SEMIGROUPS IN $\text{Sl}(2, \mathbb{R})$

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Dedicated to my father Wolfgang W. Breckner on the occasion of his 60th birthday

Abstract. We give elementary proofs (avoiding, as much as possible, any machinery of Lie theory) for the divisibility of those compression semigroups in $\text{Sl}(2, \mathbb{R})^+$ who are known to be the prototypes of the three-dimensional exponential Lie subsemigroups of $\text{Sl}(2, \mathbb{R})$.

Why this note has been written. The natural nonabelian analogues of cones in real vector spaces are the divisible closed subsemigroups of connected Lie groups, these are exactly the exponential Lie semigroups. In [4] K.H. Hofmann and W.A.F. Ruppert classify the reduced exponential Lie semigroups and show that these semigroups are built up from a few building blocks, the so-called Master Examples. In 1999 B.E. Breckner and W.A.F. Ruppert started a project devoted to the study of the topological semigroup compactifications of divisible subsemigroups of Lie groups. A first step for carrying out this project is to investigate the topological semigroup compactifications of the Master Examples. So, Breckner and Ruppert focused for the beginning on one of the Master Examples, namely the exponential Lie subsemigroups of $\text{Sl}(2, \mathbb{R})$. It has turned out, however, that for the study of the compactifications of these semigroups one needs a very detailed knowledge of general structural features of $\text{Sl}(2, \mathbb{R})$ (see [1]). We remark in passing that, using the tools introduced in [1], Breckner and Ruppert offer in [2] a fairly comprehensive study of the topological semigroup compactifications of certain subsemigroups of $\text{Sl}(2, \mathbb{R})$ (including the exponential ones). A main result of [1], with important consequences for the investigations

19
in [2], is the determination of the conjugacy classes of exponential subsemigroups of \( \text{Sl}(2, \mathbb{R}) \) (see 7.14 of [1]):

Let \( S \) be a three dimensional exponential subsemigroup of \( \text{Sl}(2, \mathbb{R}) \). Then \( S \) is conjugate to exactly one of the following semigroups:

1. \( \text{Sl}(2, \mathbb{R})^+ \),
2. \( S_1 = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R})^+ \mid a + b \geq c + d \} \),
3. \( S^1 = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R})^+ \mid a + c \geq b + d \} \),
4. \( S_\lambda^1 = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R})^+ \mid a + c \geq b + d \text{ and } a + \frac{1}{\lambda}b \geq \lambda c + d \} \), for some real \( \lambda > 0 \).

In [1] the exponentiality of the semigroups \( S_1, S^1, \) and \( S_\lambda^1 \) is shown by a typical Lie theoretical argument, involving the determination of the Lie wedges of the semigroups. Nevertheless, the exponentiality of these semigroups is of interest also from a pure algebraical point of view. To see this, recall that a closed submonoid of a connected Lie group is divisible if and only if it is an exponential Lie semigroup (cf, eg, 2.7 of [4]). Thus, a problem of own interest is to prove the divisibility of the semigroups \( S_1, S^1, \) and \( S_\lambda^1 \) by a direct, algebraical argument. The present paper offers such a proof.

**Divisible semigroups.** A semigroup \( S \) is called *divisible* if \( \forall s \in S, \forall n \in \mathbb{N}^* \exists x \in S \) such that \( x^n = s \).

**Notations.** Following [3], we write \( \text{Sl}(2, \mathbb{R})^+ \) for the semigroup of matrices with nonnegative entries in \( \text{Sl}(2, \mathbb{R}) \). For fixed positive reals \( \lambda, \mu > 0 \) we define the following subsets of \( \text{Sl}(2, \mathbb{R})^+ \):

\[
S_\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R})^+ \mid a + \frac{1}{\lambda}b \geq \lambda c + d \right\},
\]

\[
S^\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R})^+ \mid a + \frac{1}{\lambda}c \geq \lambda b + d \right\}, \text{ and } S_\lambda^\mu = S_\lambda \cap S^\mu.
\]

**The main statement.** The sets \( S_\lambda, S^\lambda, \) and \( S_\lambda^\mu \) are divisible semigroups for every \( \lambda, \mu > 0 \).
Remark. The first step in the proof of the main statement is to show that $S_\lambda, S^{2\lambda},$ and $S^{\mu\lambda}$ are indeed semigroups. For this it suffices to show that $S_\lambda$ is a semigroup, because $S^{\lambda}$ is the image of $S_\lambda$ under the anti-isomorphism sending every matrix to its transpose. That $S_\lambda$ is a semigroup is not obvious, since it cannot be seen immediately that the product of two arbitrary elements of $S_\lambda$ belongs to $S_\lambda$. So, it turned out to be very convenient to follow [1] and to represent $S_\lambda$ as a compression semigroup.

Compression semigroups. Let $S$ be a semigroup which acts on some space $X$. Then for every subset $M$ of $X$, we define the compression semigroup of $M$ in $S$ as the set

$$\text{compr}_S(M) = \{ s \in S \mid sM \subseteq M \}.$$ 

It is obvious that $\text{compr}_S(M)$ is either empty or a subsemigroup of $S$.

The set $S_\lambda$ as a compression semigroup. (cf 6.8 of [1]) Consider the natural action of $\text{SL}(2, \mathbb{R})^+$ (as a semigroup of endomorphisms of $\mathbb{R}^2$) on $\mathbb{R}^2$ and define for a fixed real $\lambda > 0$ the cone

$$C_\lambda = \{(x, y) \in \mathbb{R}^2 \mid x \geq \lambda y \geq 0\}.$$ 

The reader is invited to check by a straightforward computation that $S_\lambda$ is the compression semigroup of $C_\lambda$ in $\text{SL}(2, \mathbb{R})^+$ (see also 6.8 of [1]).

The following notion, similar to that of a compression semigroup, will be crucial for the proof of the main statement.

Almost compression semigroups. Let $S$ be a semigroup which acts on some space $X$ and consider $M, M'$ subsets of $X$ such that $M' \subseteq M$. We define the almost compression semigroup of the pair $(M, M')$ in $S$ to be the set

$$\text{alcompr}_S(M, M') = \{ s \in S \mid sM \subseteq M' \}.$$ 

It follows readily from its definition that $\text{alcompr}_S(M, M')$ is either empty or a subsemigroup of $S$.

We collect now some facts needed for the proof of the main statement.

Fact 1: The semigroup $\text{SL}(2, \mathbb{R})^+$ is divisible.
For those who are familiar with Lie theory this is a well-known result. It can be proved by direct calculation involving the formula for the exponential function exp: \( \text{sl}(2, \mathbb{R}) \to \text{Sl}(2, \mathbb{R}) \) (cf, eg, p. 416 ff. of [3]).

**Fact 2:** Let \( S \) be a divisible semigroup and \( (S_i)_{i \in I} \) a family of subsemigroups of \( S \) such that \( S \setminus S_i \) is a semigroup for every \( i \in I \). Then the intersection \( \cap_{i \in I} S_i \) is either empty or a divisible semigroup.

**Proof:** Put \( T = \cap_{i \in I} S_i \) and choose \( s \in T \) and \( n \in \mathbb{N}^* \) arbitrarily. Since \( S \) is divisible there exists \( x \in S \) such that \( x^n = s \). Then \( x \) belongs to \( T \). Otherwise the fact that \( x \not\in S_i \) for some \( i \in I \) would imply that \( s = x^n \in S \setminus S_i \), a contradiction. Thus \( T \) is a divisible subsemigroup of \( S \), if it is not empty. \( \Box \)

**Fact 3:** Let \( \lambda > 0 \). The set \( \text{Sl}(2, \mathbb{R})^+ \setminus S_\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R})^+ \mid a + \frac{1}{\lambda} b < \lambda c + d \right\} \) is a semigroup.

**Proof:** Put \( \tilde{S}_\lambda = \text{Sl}(2, \mathbb{R})^+ \setminus S_\lambda \). We prove that \( \tilde{S}_\lambda \) is an almost compression semigroup. For this consider again the natural action of \( \text{Sl}(2, \mathbb{R})^+ \) on \( \mathbb{R}^2 \) and define the sets

\[
\tilde{C}_\lambda = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x \leq \lambda y\} \setminus \{(0, 0)\}, \quad \tilde{W}_\lambda = \{(x, y) \in \tilde{C}_\lambda \mid x < \lambda y\}.
\]

We show that

(*) \( \tilde{S}_\lambda = \text{alcompr}_{\text{Sl}(2, \mathbb{R})^+} (\tilde{C}_\lambda, \tilde{W}_\lambda) \).

If \( s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R})^+ \) is such that \( s \tilde{C}_\lambda \subseteq \tilde{W}_\lambda \) then \( s \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \in \tilde{W}_\lambda \). Hence \( a + \frac{1}{\lambda} b < \lambda c + d \) or, equivalently, \( a + \frac{1}{\lambda} b < \lambda c + d \).

Conversely, if \( s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sl}(2, \mathbb{R})^+ \) is such that \( a + \frac{1}{\lambda} b < \lambda c + d \) then we observe first that \( \lambda d > b \), since multiplying the first inequality with \( d > 0 \) yields (note that \( ad = 1 + bc \))

\[
ad + \frac{1}{\lambda} bd < \lambda cd + d^2 \implies 1 + bc + \frac{1}{\lambda} bd < \lambda cd + d^2 \implies 1 < (\lambda d - b)(c + \frac{1}{\lambda} d).
\]
Pick an arbitrary \((x, y) \in \tilde{C}_\lambda\). Then there exists \(\alpha, \beta \in \mathbb{R}_+\) with \(\alpha^2 + \beta^2 \neq 0\) such that \((x, y) = \alpha(0, 1) + \beta(\lambda, 1)\). Now \[s \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \in \tilde{W}_\lambda\] as well as \[s \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} a\lambda + b \\ c\lambda + d \end{pmatrix} \in \tilde{W}_\lambda.\]

Since \(\alpha^2 + \beta^2 \neq 0\) we conclude that \(s \begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{W}_\lambda\). This proves (\(\ast\)), so \(\tilde{S}_\lambda\) is a semigroup. \(\Box\)

**Proof of the main statement:** Fact 1, Fact 2, and Fact 3 imply that \(S_\lambda\) is divisible. Since the anti-isomorphism sending every matrix to its transpose maps \(S_\lambda\) onto \(S^\lambda\), it follows that \(S^\lambda\) is also divisible and that \(\text{Sl}(2, \mathbb{R})^+ \setminus S^\lambda\) is a semigroup. Using once again Fact 2, it finally follows that \(S_\lambda^\mu = S_\lambda \cap S^\mu\) is divisible. \(\Box\)

**References**


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