LACUNARY STRONG $A$-CONVERGENCE WITH RESPECT TO A MODULUS

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Abstract. The definition of lacunary strong convergence with respect to a modulus is extended to a definition of lacunary strong $A$-convergence with respect to a modulus when $A = (a_{ik})$ is an infinite matrix of complex numbers. We study some connections between lacunary strong $A$-convergence with respect to a modulus and lacunary $A$-statistical convergence.

1. Introduction

The notion of modulus function was introduced by Nakano [11]. We recall that a modulus $f$ is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,
(iii) $f$ is increasing and
(iv) $f$ is continuous from the right at 0. It follows that $f$ must be continuous on $[0, \infty)$.

Connor [2], Esi [3], Kolk [8], Maddox [9], [10], Öztürk and Bilgin [12], Pehlivan and Fisher [13], Ruckle [14] and others used a modulus function to construct sequence spaces.

Following Freedman et al. [4], we call the sequence $\theta = (k_r)$ lacunary if it is an increasing sequence of integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r/k_{r-1}$. These notations will be used throughout the paper. The sequence space of lacunary

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strongly convergent sequences \( N_\theta \) was defined by Freedman et al. [4], as follows:

\[
N_\theta = \left\{ x = (x_i) : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s \right\}.
\]

Recently, the concept of lacunary strongly convergence was generalized by Pehlivan and Fisher [13] as below:

\[
N_\theta(f) = \left\{ x = (x_i) : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} f(|x_i - s|) = 0 \text{ for some } s \right\}.
\]

Let \( A = (a_{ik}) \) be an infinite matrix of complex numbers. We write \( Ax = (A_i(x)) = \sum_{k=1}^{\infty} a_{ik} x_k \) converges for each \( i \).

The purpose of this paper is to introduce and study a concept of lacunary strong \( A \)-convergence with respect to a modulus.

2. \( N_\theta(A, f) \) Convergence

**Definition.** Let \( A = (a_{ik}) \) be an infinite matrix of complex numbers and \( f \) be a modulus. We define

\[
N_\theta(A, f) = \left\{ x = (x_i) : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) = 0 \text{ for some } s \right\},
\]

\[
N_\theta^0(A, f) = \left\{ x = (x_i) : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} f(|A_i(x)|) = 0 \right\}.
\]

A sequence \( x = (x_k) \) is said to be lacunary strong \( A \)-convergent to a number \( s \) with respect to a modulus if there is a complex number \( s \) such that \( x \in N_\theta(A, f) \). Note that, if we put \( f(x) = x \), then \( N_\theta(A, f) = N_\theta(A) \) and \( N_\theta^0(A, f) = N_\theta^0(A) \). If \( x \in N_\theta(A) \), we say that \( x \) is lacunary strong \( A \)-convergent to \( s \). If \( x \) is lacunary strong \( A \)-convergent to the value \( s \) with respect to a modulus \( f \), then we write \( x_i \to s(N_\theta(A, f)) \).

If \( A = I \) unit matrix, we write \( N_\theta(f) \) and \( N_\theta^0(f) \) for \( N_\theta(A, f) \) and \( N_\theta^0(A, f) \), respectively. Hence \( N_\theta(f) \) is the same as the space \( N_\theta(f) \) of Pehlivan and Fisher [13].

\( N_\theta(A, f) \) and \( N_\theta^0(A, f) \) are linear spaces. We consider only \( N_\theta^0(A, f) \). Suppose that \( x, y \in N_\theta^0(A, f) \) and \( a, b \) are in \( C \), the complex numbers. Then there exist integers
$T_a$ and $T_b$ such that $|a| \leq T_a$ and $|b| \leq T_b$. We therefore have
\[
h_r^{-1} \sum_{i \in I_r} f(|aA_i(x) + bA_i(y)|) \leq T_a h_r^{-1} \sum_{i \in I_r} f(|A_i(x)|) + T_b h_r^{-1} \sum_{i \in I_r} f(|A_i(y)|).
\]
This implies that $ax + by \in N^0_\theta(A, f)$.

Now we give relation between lacunary strong A-convergence and lacunary strong A-convergence with respect to a modulus.

**Theorem 1.** Let $f$ be any modulus. Then $N_\theta(A) \subseteq N_\theta(A, f)$ and $N^0_\theta(A) \subseteq N^0_\theta(A, f)$.

**Proof.** We consider $N_\theta(A) \subseteq N_\theta(A, f)$ only. Let $x \in N_\theta(A)$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every $u$ with $0 \leq u \leq \delta$. We can write
\[
h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) = h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) + h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|)
\]
where the first summation is over $|A_i(x) - s| \leq \delta$ and the second over $|A_i(x) - s| > \delta$. By definition of $f$, we have
\[
h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \leq \varepsilon + 2f(1)\delta^{-1} h_r^{-1} \sum_{i \in I_r} |A_i(x) - s|.
\]
Therefore $x \in N_\theta(A, f)$.

**Theorem 2.** Let $f$ be any modulus. If \( \lim_{t \to \infty} \frac{f(t)}{t} = \beta > 0 \), then $N_\theta(A) = N_\theta(A, f)$.

**Proof.** If \( \lim_{t \to \infty} \frac{f(t)}{t} = \beta > 0 \), then $f(t) \geq \beta t$ for all $t > 0$. Let $x \in N_\theta(A, f)$.

Clearly,
\[
h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \geq h_r^{-1} \sum_{i \in I_r} \beta |A_i(x) - s| = \beta h_r^{-1} \sum_{i \in I_r} |A_i(x) - s|,
\]
therefore $x \in N_\theta(A)$. By using Theorem 1 the proof is complete.

We now give an example to show that $N_\theta(A) \neq N_\theta(A, f)$ in the case when $\beta = 0$. Consider $A = I$ and the modulus $f(x) = \sqrt{x}$. In the case $\beta = 0$, define $x_i$ to be $h_r$ at the first term in $I_r$ for every $r$ and $x_i = 0$ otherwise. Then we have
\[
h_r^{-1} \sum_{i \in I_r} f(|A_i(x)|) = h_r^{-1} \sum_{i \in I_r} \sqrt{|x_i|} = h_r^{-1} \sqrt{|h_r|} \to 0 \text{ as } r \to \infty
\]
and so $x \in N_\theta(A, f)$. But $h_r^{-1} \sum_{i \in I_r} |A_i(x)| = h_r^{-1} \sum_{i \in I_r} |x_i| = h_r^{-1} h_r \to 1$ as $r \to \infty$ and so $x \notin N_\theta(A)$.

**Theorem 3.** Let $f$ be any modulus. Then
(i) For $\liminf q_r > 1$ we have $w(A, f) \subseteq N_0(A, f)$.

(ii) For $\limsup q_r < \infty$ we have $N_0(A, f) \subseteq w(A, f)$.

(iii) $w(A, f) = N_0(A, f)$ is $1 \succ \liminf q_r \leq \limsup q_r < \infty$.

where $w(A, f) = \left\{ x = (x_1) : \liminf_{n \to \infty} n^{-1} \sum_{i=1}^{n} f(|A_i(x) - s|) = 0 \text{ for some } s \right\}$ (see, Esi [3]).

**Proof.** (i) Let $x \in w(A, f)$ and $\liminf q_r > 1$. There exist $\delta > 0$ such that $q_r = (k_r/k_{r-1}) \geq 1 + \delta$ for sufficiently large $r$. We have, for sufficiently large $r$, that $(h_r/k_r) \geq \delta/(1 + \delta)$ and $(k_r/h_r) \leq (1 + \delta)/\delta$. Then

$$k_r^{-1} \sum_{i=1}^{k_r} f(|A_i(x) - s|) \geq k_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) = (h_r/k_r) h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) \geq \delta/(1 + \delta) h_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|)$$

which yields that $x \in N_0(A, f)$.

(ii) If $\limsup q_r < \infty$ then there exists $K > 0$ such that $q_r < K$ for every $r$. Now suppose that $x \in N_0(A, f)$ and $\varepsilon > 0$. There exists $m_0$ such that for every $m \geq m_0$,

$$H_m = h_m^{-1} \sum_{i \in I_m} f(|A_i(x) - x|) < \varepsilon.$$ We can also find $T > 0$ such that $H_m \leq T$ for all $m$. Let $n$ be any integer with $k_r \geq n \succ k_{r-1}$. Now write

$$n^{-1} \sum_{i=1}^{n} f(|A_i(x) - s|) \leq k_r^{-1} \sum_{i=1}^{k_r} f(|A_i(x) - s|)$$

$$= k_r^{-1} \left( \sum_{m=1}^{m_0} \sum_{i \in I_m} f(|A_i(x) - s|) + \sum_{m=m_0+1}^{k_r} \sum_{i \in I_m} f(|A_i(x) - s|) \right)$$

$$= k_r^{-1} \sum_{m=1}^{m_0} \sum_{i \in I_m} f(|A_i(x) - s|) + k_r^{-1} \sum_{m=m_0+1}^{k_r} \sum_{i \in I_m} f(|A_i(x) - s|)$$

$$\leq k_r^{-1} \sum_{m=1}^{m_0} \sum_{i \in I_m} f(|A_i(x) - s|) + \varepsilon (k_r - k_{m_0}) k_r^{-1}$$

$$= k_r^{-1} (h_1 H_1 + h_2 H_2 + \cdots + h_{m_0} H_{m_0}) + \varepsilon (k_r - k_{m_0}) k_r^{-1}$$

$$\leq k_r^{-1} \left( \sup_{1 \leq i \leq m_0} H_i k_{m_0} \right) + \varepsilon K \leq k_r^{-1} k_{m_0} T + \varepsilon K.$$
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from which we deduce that \( x \in w(A, f) \). (iii) follows from (i) and (ii).

The next result follows from Theorem 2 and 3.

**Theorem 4.** Let \( f \) be any modulus. If \( \lim_{t \to \infty} \frac{f(t)}{t} = \beta \gg 0 \) and \( l \ll \lim \inf r \leq \lim \sup r \ll \infty \), then \( N_\theta(A) = w(A, f) \).

3. Lacunary A-statistical convergence

The notation of statistical convergence was given in earlier works [1], [4], [6], [15] and [16]. Recently, Fridy and Orhan [7] introduced the concept of lacunary statistical convergence:

Let \( \theta \) be a lacunary sequence. Then a sequence \( x = (x_k) \) is said to be lacunary statistically convergent to a number \( s \) if for every \( \varepsilon \gg 0 \), \( \lim_{r \to \infty} h_r^{-1}|K_\theta(\varepsilon)| = 0 \), where \( |K_\theta(\varepsilon)| \) denotes the number of elements in \( K_\theta(\varepsilon) = \{i \in I_r : |x_i - s| \geq \varepsilon \} \). The set of all lacunary statistical convergent sequences is denoted by \( S_\theta \).

Let \( A = (a_{ik}) \) be an infinire matrix of complex numbers. Then a sequence \( x = (x_k) \) is said to be lacunary A-statistically convergent to a number \( s \) if for every \( \varepsilon \gg 0 \), \( \lim_{r \to \infty} h_r^{-1}|KA_\theta(\varepsilon)| = 0 \), where \( |KA_\theta(\varepsilon)| \) denotes the number of element in \( KA_\theta(\varepsilon) = \{i \in I : |A_i(x) - s| \geq \varepsilon \} \). The set of all lacunary A-statistical convergent sequences is denoted by \( S_\theta(A) \).

The following Theorem gives the relation between of the lacunary A-statistical convergence and lacunary strongly A-convergence.

Let \( I_1 = \{i \in I_r : |A_i(x) - s| \geq \varepsilon \} = KA_\theta(\varepsilon) \) and \( I_2 = \{i \in I_r : |A_i(x) - s| \ll \varepsilon \} \).

**Theorem 5.** Let \( A \) be a limitation method, then

(i) \( x_i \to s(N_\theta(A)) \) implies \( x_i \to s(S_\theta(A)) \).

(ii) \( x \) is bounded and \( x_i \to s(S_\theta(A)) \) implies \( x_i \to s(N_\theta(A)) \).

(iii) \( S_\theta(A) = N_\theta(A) \) is \( x \) is bounded.

**Proof.** (i) If \( \varepsilon \gg 0 \) and \( x_i \to s(N_\theta(A)) \) we can write

\[
h_r^{-1} \sum_{i \in I_r} |A_i(x) - s| \geq h_r^{-1}|KA_\theta(\varepsilon)|\varepsilon.
\]

It follows that \( x_i \to s(S_\theta(A)) \). Note that in this part of the proof we do not need the limitation method of \( A \).
(ii) Suppose that $x$ is lacunary A-statistical convergent to $s$. Since $x$ is bounded and $A$ is limitation method, there is a constant $T > 0$ such that $|A_i(x) - s| \leq T$ for all $i$. Therefore we have, for every $\varepsilon > 0$, that

$$h^{-1}_r \sum_{i \in I_r} |A_i(x) - s| \leq h^{-1}_r \sum_{i \in I_r} |A_i(x) - s| + h^{-1}_r \sum_{i \in I_r^2} |A_i(x) - s| \leq T h^{-1}_r |KA_\theta(\varepsilon)| + \varepsilon.$$

Taking the limit as $\varepsilon \to 0$, the result follows. (iii) follows from (i) and (ii).

Now we give the relation between of the lacunary A-statistical convergence and lacunary strongly A-convergence with respect to modulus.

**Theorem 6.** (i) For any modulus $f$, $x_i \to s(N_\theta(A,f))$ implies $x_i \to s(S_\theta(A))$.

(ii) $f$ is bounded and $x_i \to s(S_\theta(A))$ imply $x_i \to s(N_\theta(A,f))$.

(iii) $S_\theta(A) = N_\theta(A,f)$ if $f$ is bounded.

**Proof.** (i) Let $f$ be any modulus. If $\varepsilon > 0$ and $x_i \to s(N_\theta(A,f))$ we can write

$$h^{-1}_r \sum_{i \in I_r} f(|A_i(x) - s|) \geq h^{-1}_r \sum_{i \in I_r} f(|A_i(x) - s|) \geq h^{-1}_r |KA_\theta(\varepsilon)| f(\varepsilon).$$

It follows that $x_i \to s(S_\theta(A))$.

(ii) Suppose that $f$ is bounded. Since $f$ is bounded, there exists an integer $T$ such that $f(x) \leq T$ for all $x \geq 0$. We see that

$$h^{-1}_r \sum_{i \in I_r} f(|A_i(x) - s|) \leq h^{-1}_r \sum_{i \in I_r} f(|A_i(x) - s|) + h^{-1}_r \sum_{i \in I_r^2} f(|A_i(x) - s|) \leq T h^{-1}_r |KA_\theta(\varepsilon)| + f(\varepsilon).$$

Since $f$ is continuous and $x_i \to s(S_\theta(A))$, it follows from $\varepsilon \to 0$ that $x_i \to s(N_\theta(A,f))$. (ii) follows from (i) and (ii).

As an example to show that $S_\theta(A) \neq N_\theta(A,f)$ when $f$ is unbounded, consider $A = I$. Since $f$ is unbounded, there exists a positive sequence $0 < y_1 < y_2 < \ldots$ such that $f(y_i) \geq h_i$. Define the sequence $x = (x_i)$ by putting $x_{ki} = y_i$ for $i = 1, 2, \ldots$ and $x_i = 0$ otherwise. We have $x \in S_\theta(A)$, but $x \not\in N_\theta(A,f)$.

Finally we consider the case when $x_k \to s$ implies $x_k \to s(N_\theta(A,f))$.

**Lemma 7.** ([6]) If $\liminf q_r > 1$ then $x_i \to s(S)$ implies $x_i \to s(S_\theta)$. 

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Theorem 8. Let $\liminf q_r \succ 1$, $A$ is regular and $f$ is bounded. Then $x_i \to s$ implies $x_i \to s(\mathcal{N}_\theta(A,f))$.

Proof. Let $x_i \to s$. By regularity of $A$ and definition of statistical convergence we have $A_i(x) \to s(S)$. Since $\liminf q_r \succ 1$ it follows lemma 7 that $A_i(x) \to s(S_\theta)$ i.e. $x_i \to s(S_\theta(A))$. Thus, using Theorem 6, we have $x_i \to s(\mathcal{N}_\theta(A,f))$.

References


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