ON A CERTAIN FAMILIES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. We introduce the subclass \( T_j(n, m, \lambda, \alpha) \) of analytic functions with negative coefficients defined by Salagean operators \( D^n \) and \( D^{n+m} \). In this paper we give some properties of functions in the class \( T_j(n, m, \lambda, \alpha) \) and obtain numerous sharp results including (for example) coefficient estimates, distortion theorems, closure theorems and modified Hadamard products of several functions belonging to the class \( T_j(n, m, \lambda, \alpha) \). We also obtain radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class \( T_j(n, m, \lambda, \alpha) \) and consider integral operators associated with functions belonging to the class \( T_j(n, m, \lambda, \alpha) \).

1. Introduction

Let \( A(j) \) denote the class of functions of the form

\[
f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N} = \{1, 2, \ldots \}),
\]

which are analytic in the unit disc \( U = \{ z : |z| < 1 \} \). For a function \( f(z) \) in \( A(j) \), we define

\[
D^0 f(z) = f(z),
\]

\[
D^1 f(z) = D f(z) = zf'(z)
\]

and

\[
D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}).
\]

The differential operator \( D^n \) was introduced by Salagean [5]. With the help of the differential operator \( D^n \), we say that a function \( f(z) \) belonging to \( A(j) \) is in

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the class \( S_j(n, m, \lambda, \alpha) \) if and only if
\[
\text{Re} \left\{ \frac{(1 - \lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1 - \lambda)D^n f(z) + \lambda D^{n+m} f(z)} \right\} > \alpha \quad (n, m \in N_0 = N \cup \{0\}) \quad (1.5)
\]
for some \( \alpha (0 \leq \alpha < 1) \) and \( \lambda (0 \leq \lambda \leq 1) \), and for all \( z \in U \). The operator \( D^{n+m} \) was studied by Sekine [7] and Aouf and Salagean [2].

Let \( T(j) \) denote the subclass of \( A(j) \) consisting of functions of the form
\[
f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0; \; j \in N). \quad (1.6)
\]

Further, we define the class \( T_j(n, m, \lambda, \alpha) \) by
\[
T_j(n, m, \lambda, \alpha) = S_j(n, m, \lambda, \alpha) \cap T(j). \quad (1.7)
\]

We note that by specializing the parameters \( j, n, m, \lambda \) and \( \alpha \), we obtain the following subclasses studied by various authors:

(i) \( T_j(n, 1, \lambda, \alpha) = P(j, \lambda, \alpha, n) \), \( T_j(n, m, 0, \alpha) = P(j, \alpha, n) \) and \( T_j(n, 1, 1, \alpha) = P(j, \alpha, n + 1) \) (Aouf and Srivastava [3]);

(ii) \( T_j(0, 1, \lambda, \alpha) = P(j, \lambda, \alpha) \) (Altintas [1]);

(iii) \( T_j(0, 0, 0, \alpha) = T_\alpha(j) \) and \( T_j(0, 1, 1, \alpha) = T_j(1, 0, 1, \alpha) = C_\alpha(j) \) (Chatterjea [4] and Srivastava et al. [9]);

(iv) \( T_j(n, 1, 1, \alpha) = T_j(n, m, \alpha) \), where \( T_j(n, m, \alpha) \) represents the class of functions \( f(z) \in T(j) \) satisfying the condition
\[
\text{Re} \left\{ \frac{z(D^{n+m} f(z))'}{D^{n+m} f(z)} \right\} > \alpha \quad (n, m \in N_0; \; 0 \leq \alpha < 1; \; z \in U); \quad (1.8)
\]

(iv) \( T_1(0, 0, 0, \alpha) = T^*(\alpha) \) and \( T_1(0, 1, 1, \alpha) = T_1(1, 0, 1, \alpha) = C(\alpha) \) (Silverman [8]).

2. Coefficient estimates and other properties of the class \( T_j(n, m, \lambda, \alpha) \)

**Theorem 1.** Let the function \( f(z) \) be defined by (1.6). Then \( f(z) \in T_j(n, m, \lambda, \alpha) \) if and only if
\[
\sum_{k=j+1}^{\infty} k^n (k - \alpha)[1 + (k^m - 1)\lambda] a_k \leq 1 - \alpha. \quad (2.1)
\]

The result is sharp.
Proof. Assume that the inequality (2.1) holds true. Then we find that

\[
\left| \frac{(1 - \lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1 - \lambda)D^n f(z) + \lambda D^{n+m} f(z)} - 1 \right| \leq
\]

\[
\sum_{k=j+1}^{\infty} k^n (k - 1)|1 + (k^m - 1)\lambda|a_k|z|^{k-1}
\]

\[
1 - \sum_{k=j+1}^{\infty} k^n |1 + (k^m - 1)\lambda|a_k|z|^{k-1}
\]

\[
\leq k^n (k - 1)|1 + (k^m - 1)\lambda|a_k
\]

\[
1 - \sum_{k=j+1}^{\infty} k^n |1 + (k^m - 1)\lambda|a_k
\]

\[
\leq 1 - \alpha.
\]

This shows that the values of the function

\[
\Phi(z) = \frac{(1 - \lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1 - \lambda)D^n f(z) + \lambda D^{n+m} f(z)}
\]

lie in a circle which is centered at \( w = 1 \) and whose radius is \( 1 - \alpha \). Hence \( f(z) \) satisfies the condition (1.5).

Conversely, assume that the function \( f(z) \) is in the class \( T_j(n, m, \lambda, \alpha) \). Then we have

\[
\text{Re} \left\{ \frac{(1 - \lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1 - \lambda)D^n f(z) + \lambda D^{n+m} f(z)} \right\} =
\]

\[
\text{Re} \left\{ 1 - \sum_{k=j+1}^{\infty} k^{n+1}[1 + (k^m - 1)\lambda]a_k z^{k-1} \right\}
\]

\[
1 - \sum_{k=j+1}^{\infty} k^n |1 + (k^m - 1)\lambda|a_k |z|^{k-1}
\]

\[
> \alpha, \quad \text{for some } \alpha (0 \leq \alpha < 1), \lambda (0 \leq \lambda \leq 1), n, m \in N_0 \text{ and for all } z \in U.
\]

Choose values of \( z \) on the real axis so that \( \Phi(z) \) given by (2.2) is real. Upon clearing the denominator in (2.3) and letting \( z \to 1^- \) through real values, we can see that

\[
1 - \sum_{k=j+1}^{\infty} k^{n+1}[1 + (k^m - 1)\lambda]a_k \geq \alpha \left\{ 1 - \sum_{k=j+1}^{\infty} k^n[1 + (k^m - 1)\lambda]a_k \right\}.
\]

(2.4)

Thus we have the inequality (2.1).

Finally, the function \( f(z) \) given by

\[
f(z) = z - \frac{1 - \alpha}{k^n(k - \alpha)[1 + (k^m - 1)\lambda]} z^k \quad (k \geq j + 1; \ j \in N)
\]

(2.5)
is an extremal function for the assertion of Theorem 1.

**Corollary 1.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n, m, \lambda, \alpha) \). Then

\[
a_k \leq \frac{1 - \alpha}{kn(k - \alpha)[1 + (k^m - 1)\lambda]} \quad (k \geq j + 1).
\]

(2.6)

The equality in (2.6) is attained for the function \( f(z) \) given by (2.5).

**Theorem 2.** Let \( 0 \leq \alpha_1 \leq \alpha_2 < 1 \), \( 0 \leq \lambda \leq 1 \), \( j \in \mathbb{N} \) and \( n, m \in \mathbb{N}_0 \). Then

\[
T_j(n, m, \lambda, \alpha_1) \supseteq T_j(n, m, \lambda, \alpha_2).
\]

(2.7)

**Proof.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n, m, \lambda, \alpha_2) \) and let \( \alpha_1 = \alpha_2 - \delta \). Then, by Theorem 1, we have

\[
\sum_{k=j+1}^{\infty} k^n(k - \alpha)[1 + (k^m - 1)\lambda]a_k \leq 1 - \alpha_2
\]

(2.8)

and

\[
\sum_{k=j+1}^{\infty} k^n[1 + (k^m - 1)\lambda]a_k \leq \frac{1 - \alpha_2}{f + 1 - \alpha_2} < 1.
\]

(2.9)

Consequently,

\[
\sum_{k=j+1}^{\infty} k^n(k - \alpha_1)[1 + (k^m - 1)\lambda]a_k = \sum_{k=j+1}^{\infty} k^n(k - \alpha_2)[1 + (k^m - 1)\lambda]a_k +
\]

\[
+ \delta \sum_{k=j+1}^{\infty} k^n[1 + (k^m - 1)\lambda]a_k \leq 1 - \alpha_1.
\]

(2.10)

This completes the proof of Theorem 2 with the aid of Theorem 1.

**Theorem 3.** Let \( 0 \leq \alpha < 1 \), \( 0 \leq \lambda_1 \leq \lambda_2 \leq 1 \), \( j \in \mathbb{N} \) and \( n, m \in \mathbb{N}_0 \). Then

\[
T_j(n, m, \lambda_1, \alpha) \supseteq T_j(n, m, \lambda_2, \alpha).
\]

(2.11)

**Proof.** It follows from Theorem 1 that

\[
\sum_{k=j+1}^{\infty} k^n(k - \alpha)[1 + (k^m - 1)\lambda_1]a_k \leq \sum_{k=j+1}^{\infty} k^n(k - \alpha)[1 + (k^m - 1)\lambda_2]a_k \leq 1 - \alpha
\]

for \( f(z) \in T_j(n, m, \lambda_2, \alpha) \).

**Theorem 4.** For \( 0 \leq \alpha < 1 \), \( 0 \leq \lambda \leq 1 \), \( j \in \mathbb{N} \) and \( n, m \in \mathbb{N}_0 \),

\[
T_j(n + 1, m, \lambda, \alpha) \subseteq T_j(n, m, \lambda, \alpha).
\]

(2.12)
The proof of Theorem 4 follows also from Theorem 1.

3. Growth and distortion theorems

**Theorem 5.** Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then for $|z| = r < 1$,

$$|D^i f(z)| \geq r - \frac{1 - \alpha}{(j+1)^{n-i}(j+1-\alpha)[1+((j+1)^m-1)\lambda]} r^{j+1} \quad (3.1)$$

and

$$|D^i f(z)| \leq r + \frac{1 - \alpha}{(j+1)^{n-i}(j+1-\alpha)[1+((j+1)^m-1)\lambda]} r^{j+1} \quad (3.2)$$

for $z \in U$ and $0 \leq i \leq n$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1 - \alpha}{(j+1)^{n}(j+1-\alpha)[1+((j+1)^m-1)\lambda]} z^{j+1} \quad (z = \pm r). \quad (3.3)$$

**Proof.** Note that $f(z) \in T_j(n, m, \lambda, \alpha)$ if and only if

$$D^i f(z) \in T_j(n-i, m, \lambda, \alpha)$$

and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \quad (3.4)$$

By Theorem 1, we know that

$$(j+1)^{n-i}(j+1-\alpha)[1+((j+1)^m-1)\lambda] \sum_{k=j+1}^{\infty} k^i a_k \leq$$

$$\leq \sum_{k=j+1}^{\infty} k^n (k-\alpha)[1+(k^m-1)\lambda] a_k \leq 1 - \alpha, \quad (3.5)$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i a_k \leq \frac{1 - \alpha}{(j+1)^{n-i}(j+1-\alpha)[1+((j+1)^m-1)\lambda]} \quad (3.6)$$

The assertions (3.1) and (3.2) of Theorem 5 would now follow readily from (3.4) and (3.6).
Finally, we note that the equalities in (3.1) and (3.2) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{1 - \alpha}{(j + 1)^{n-i}(j + 1 - \alpha)[1 + [(j + 1)^m - 1]|\lambda|]} z^{j+1}. \quad (3.7)$$

This completes the proof of Theorem 5.

**Corollary 2.** Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then, for $|z| = r < 1$,

$$|f(z)| \geq r - \frac{1 - \alpha}{(j + 1)^n(j + 1 - \alpha)[1 + [(j + 1)^m - 1]|\lambda|]} r^{j+1} \quad (3.8)$$

and

$$|f(z)| \leq r + \frac{1 - \alpha}{(j + 1)^n(j + 1 - \alpha)[1 + [(j + 1)^m - 1]|\lambda|]} r^{j+1}. \quad (3.9)$$

The equalities in (3.8) and (3.9) are attained for the function $f(z)$ given by (3.3).

**Proof.** Taking $i = 0$ in Theorem 5, we immediately obtain (3.8) and (3.9).

**Corollary 3.** Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then for $|z| = r < 1$,

$$|f'(z)| \geq \frac{1 - \alpha}{(j + 1)^n(j + 1 - \alpha)[1 + [(j + 1)^m - 1]|\lambda|]} r^j \quad (3.10)$$

and

$$|f'(z)| \leq 1 + \frac{1 - \alpha}{(j + 1)^n(j + 1 - \alpha)[1 + [(j + 1)^m - 1]|\lambda|]} r^j \quad (z \in U). \quad (3.11)$$

The equalities in (3.10) and (3.11) are attained for the function $f(z)$ given by (3.3).

**Proof.** Setting $i = 1$ in Theorem 5, and making use of the definition (1.3), we arrive at Corollary 3.

4. **Convex linear combinations**

In this section, we shall prove that the class $T_j(n, m, \lambda, \alpha)$ is closed under convex linear combinations.

**Theorem 6.** $T_j(n, m, \lambda, \alpha)$ is a convex set.

**Proof.** Let the functions

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0; \ v = 1, 2) \quad (4.1)$$
be in the class $T_j(n, m, \lambda, \alpha)$. It is sufficient to show that the function $h(z)$ defined by
\[ h(z) = \mu f_1(z) + (1 - \mu) f_2(z) \quad (0 \leq \mu \leq 1) \tag{4.2} \]
is also in the class $T_j(n, m, \lambda, \alpha)$. Since, for $0 \leq \mu \leq 1$,
\[ h(z) = z - \sum_{k=j+1}^{\infty} [\mu a_{k,1} + (1 - \mu) a_{k,2}] z^k, \tag{4.3} \]
with the aid of Theorem 1, we have
\[ \sum_{k=j+1}^{\infty} k^n (k-\alpha) [1 + (k^m - 1) \lambda] \mu_k z^k \leq 1 - \alpha, \tag{4.4} \]
which implies that $f(z) \in T_j(n, m, \lambda, \alpha)$. Hence $T_j(n, m, \lambda, \alpha)$ is a convex set.

**Theorem 7.** Let
\[ f_j(z) = z \tag{4.5} \]
and
\[ f_k(z) = z - \frac{1 - \alpha}{k^n(k-\alpha) [1 + (k^m - 1) \lambda]} z^k \quad (k \geq j + 1; \ n, m \in \mathbb{N}_0) \tag{4.6} \]
for $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$. Then $f(z)$ is in the class $T_j(n, m, \lambda, \alpha)$ if and only if it can be expressed in the form
\[ f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z), \tag{4.7} \]
where
\[ \mu_k \geq 0 \ (k \geq j) \quad \text{and} \quad \sum_{k=j}^{\infty} \mu_k = 1. \tag{4.8} \]

**Proof.** Assume that
\[ f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z) = \]
\[ = z - \sum_{k=j+1}^{\infty} \frac{1 - \alpha}{k^n (k-\alpha) [1 + (k^m - 1) \lambda]} \mu_k z^k. \tag{4.9} \]
Then it follows that
\[ \sum_{k=j+1}^{\infty} \frac{k^n (k-\alpha) [1 + (k^m - 1) \lambda]}{1 - \alpha} \cdot \frac{1 - \alpha}{k^n (k-\alpha) [1 + (k^m - 1) \lambda]} \mu_k = \]
\[ = \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \leq 1. \tag{4.10} \]
So, by Theorem 1, \( f(z) \in T_j(n, m, \lambda, \alpha) \).

Conversely, assume that the function \( f(z) \) defined by (1.6) belongs to the class \( T_j(n, m, \lambda, \alpha) \). Then
\[
 a_k \leq \frac{1 - \alpha}{k^n(k - \alpha)[1 + (k^m - 1)\lambda]} (k \geq j + 1; n, m \in N_0).
\] (4.11)

Setting
\[
 \mu_k = \frac{k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha} a_k (k \geq j + 1; n, m \in N_0)
\] (4.12)
and
\[
 \mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k,
\] (4.13)
we can see that \( f(z) \) can be expressed in the form (4.7). This completes the proof of Theorem 7.

5. Radii of close-to-convexity, starlikeness, and convexity

**Theorem 8.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n, m, \lambda, \alpha) \). Then \( f(z) \) is close-to-convex of order \( \rho \) (0 \( \leq \rho < 1 \)) in \(|z| < r_1 \), where
\[
 r_1 = r_1(n, m, \lambda, \alpha, \rho) = \inf_k \left[ \frac{(1 - \rho)k^{n-1}(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha} \right]^{\frac{1}{k-1}} (k \geq j + 1).
\] (5.1)
The result is sharp, the extremal function \( f(z) \) begin given by (2.5).

**Proof.** We must show that
\[
 |f'(z) - 1| \leq 1 - \rho \quad \text{for} |z| < r_1(n, m, \lambda, \alpha, \rho),
\]
where \( r_1(n, m, \lambda, \alpha, \rho) \) is given by (5.1). Indeed we find from the definition (1.6) that
\[
 |f'(z) - 1| \leq \sum_{k=j+1}^{\infty} ka_k |z|^{k-1}.
\]
Thus
\[
 |f'(z) - 1| \leq 1 - \rho
\]
if
\[
 \sum_{k=j+1}^{\infty} \left( \frac{k}{1 - \rho} \right) a_k |z|^{k-1} \leq 1.
\] (5.2)
But, by Theorem 1, (5.2) will be true if
\[
\left( \frac{k}{1 - \rho} \right) |z|^{k-1} \leq \frac{k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha},
\]
that is, if
\[
|z| \leq \left[ \frac{(1 - \rho)k^{n-1}(k - \alpha)[1 + (k^m - 1)\lambda]}{k - \rho(1 - \alpha)} \right]^{1/1-\alpha} (k \geq j + 1). \tag{5.3}
\]
Theorem 8 follows easily from (5.3).

**Theorem 9.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n, m, \lambda, \alpha) \). Then \( f(z) \) is starlike of order \( \rho \) (\( 0 \leq \rho < 1 \)) in \( |z| < r_2 \), where
\[
r_2 = r_2(n, m, \lambda, \alpha, \rho) = \inf f \left[ \frac{(1 - \rho)k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{(k - \rho)(1 - \alpha)} \right]^{1/1-\alpha} (k \geq j + 1). \tag{5.4}
\]
The result is sharp, with the extremal function \( f(z) \) given by (2.5).

**Proof.** It is sufficient to show that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \text{ for } |z| < r_2(n, m, \lambda, \alpha, \rho),
\]
where \( r_2(n, m, \lambda, \alpha, \rho) \) is given by (5.4). Indeed we find, again from the definition (1.6), that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{k=j+1}^{\infty} (k-1)a_k|z|^{k-1}}{1 - \sum_{k=j+1}^{\infty} a_k|z|^{k-1}}.
\]
Thus
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho
\]
if
\[
\sum_{k=j+1}^{\infty} \left( \frac{k - \rho}{1 - \rho} \right) a_k|z|^{k-1} \leq 1. \tag{5.5}
\]
But, by Theorem 1, (5.5) will be if
\[
\left( \frac{k - \rho}{1 - \rho} \right) |z|^{k-1} \leq \frac{k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha},
\]
that is, if
\[
|z| \leq \left[ \frac{(1 - \rho)k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{(k - \rho)(1 - \alpha)} \right]^{1/1-\alpha} (k \geq j + 1). \tag{5.6}
\]
Theorem 9 follows easily from (5.6).
Corollary 4. Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n, m, \lambda, \alpha) \). Then \( f(z) \) is convex of order \( \rho \) (\( 0 \leq \rho < 1 \)) in \( |z| < r_3 \), where
\[
r_3 = r_3(n, m, \lambda, \alpha, \rho) = \inf_k \left[ \frac{(1-\rho)k^{n-1}(k-\alpha)[1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)} \right]^{1/\rho} (k \geq j + 1).
\] (5.7)

The result is sharp, with the extremal function \( f(z) \) given by (2.5).

6. Modified Hadamard products

Let the functions \( f_v(z) \) (\( v = 1, 2 \)) be defined by (4.1). The modified Hadamard product of \( f_1(z) \) and \( f_2(z) \) is defined by
\[
f_1 * f_2(z) = z - \sum_{k=j+1}^{\infty} a_{k,1}a_{k,2}z^k.
\] (6.1)

Theorem 10. Let each of the functions \( f_v(z) \) (\( v = 1, 2 \)) defined by (4.1) be in the class \( T_j(n, m, \lambda, \alpha) \). Then
\[
f_1 * f_2(z) \in T_j(n, m, \lambda, \beta(j, n, m, \lambda, \alpha)),
\]
where
\[
\beta(j, n, m, \lambda, \alpha) = 1 - \frac{j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+\lambda(j+1)^m-1]-(1-\alpha)^2}.
\] (6.2)

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to find the largest \( \beta = \beta(j, n, m, \lambda, \alpha) \) such that
\[
\sum_{k=j+1}^{\infty} \frac{k^n(k-\beta)[1+(k^m-1)\lambda]}{1-\beta} a_{k,1}a_{k,2} \leq 1.
\] (6.3)

Since
\[
\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,1} \leq 1
\] (6.4)
and
\[
\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,2} \leq 1,
\] (6.5)

by the Cauchy-Schwarz inequality, we have
\[
\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \sqrt{a_{k,1}a_{k,2}} \leq 1.
\] (6.6)
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Thus it is sufficient to show that

$$\frac{k^n(k - \beta)[1 + (k^m - 1)\lambda]}{1 - \beta} \leq \frac{k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha} \sqrt{a_{k,1}a_{k,2}} \quad (k \geq j + 1),$$

(6.7)

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(k - \alpha)(1 - \beta)}{(k - \beta)(1 - \alpha)} \quad (k \geq j + 1).$$

(6.8)

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{1 - \alpha}{k^n(k - \alpha)[1 + (k^m - 1)\lambda]} \quad (k \geq j + 1).$$

(6.9)

Consequently, we need only to prove that

$$\frac{1 - \alpha}{k^n(k - \alpha)[1 + (k^m - 1)\lambda]} \leq \frac{(k - \alpha)(1 - \beta)}{(k - \beta)(1 - \alpha)} \quad (k \geq j + 1),$$

(6.10)

or, equivalently, that

$$\beta \leq 1 - \frac{(k - 1)(1 - \alpha)^2}{k^n(k - \alpha)^2[1 + (k^m - 1)\lambda] - (1 - \alpha)^2} \quad (k \geq j + 1).$$

(6.11)

Since

$$A(k) = 1 - \frac{(k - 1)(1 - \alpha)^2}{k^n(k - \alpha)^2[1 + (k^m - 1)\lambda] - (1 - \alpha)^2}$$

is an increasing function of $k$ ($k \geq j + 1$), letting $k = j + 1$ in (6.12) we obtain

$$\beta \leq A(j + 1) = \frac{j(1 - \alpha)^2}{(j + 1)^n(j + 1 - \alpha)[1 + [(j + 1)^m - 1]\lambda] - (1 - \alpha)^2},$$

(6.13)

which proves the main assertion of Theorem 10.

Finally, by taking the functions

$$f_v(z) = z - \frac{1 - \alpha}{(j + 1)^n(j + 1 - \alpha)[1 + [(j + 1)^m - 1]\lambda]}z^{j+1} \quad (v = 1, 2),$$

(6.14)

we can see that the result is sharp.

**Theorem 11.** Let $f_1(z) \in T_j(n, m, \lambda, \alpha)$ and $f_2(z) \in T_j(n, m, \lambda, \gamma)$. then

$$f_1 \ast f_2(z) \in T_j(n, m, \lambda, \xi(j, n, m, \lambda, \alpha, \gamma)),$$

where

$$\xi(j, n, m, \lambda, \alpha, \gamma) =$$

$$= 1 - \frac{j(1 - \alpha)(1 - \gamma)}{(j + 1)^n(j + 1 - \alpha)(j + 1 - \gamma)[1 + [(j + 1)^m - 1]\lambda] - (1 - \alpha)(1 - \gamma)}.$$
The result is best possible for the functions

\[ f_1(z) = z - \frac{1 - \alpha}{(j + 1)^n(j + 1 - \alpha)[1 + [(j + 1)^m - 1] \lambda]^z} z^{j+1} \] \quad (6.16)

and

\[ f_2(z) = z - \frac{1 - \gamma}{(j + 1)^n(j + 1 - \gamma)[1 + [(j + 1)^m - 1] \lambda]^z} z^{j+1}. \] \quad (6.17)

**Proof.** Proceeding as in the proof of Theorem 10, we get

\[ \xi \leq 1 - \frac{(k - 1)(1 - \alpha)(1 - \gamma)}{k^n(k - \alpha)(k - \gamma)[1 + (k^m - 1) \lambda] - (1 - \alpha)(1 - \gamma)} \quad (k \geq j + 1). \] \quad (6.18)

Since the right hand side of (6.18) is an increasing function of \( k \), setting \( k = j + 1 \) in (6.18) we obtain (6.15). This completes the proof of Theorem 11.

**Corollary 5.** Let the functions \( f_v(z) \) defined by

\[ f_v(z) = z - \sum_{k=j+1}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0, \ v = 1, 2, 3) \] \quad (6.19)

be in the class \( T_j(n, m, \lambda, \alpha) \). Then

\[ f_1 * f_2 * f_3(z) \in T_j(n, m, \lambda, \delta(j, n, m, \lambda, \alpha)), \]

where

\[ \delta(j, n, m, \lambda, \alpha) = 1 - \frac{j(1 - \alpha)^3}{(j + 1)^2n(j + 1 - \alpha)^3[1 + [(j + 1)^m - 1] \lambda]^2 - (1 - \alpha)^3}. \] \quad (6.20)

The result is best possible for the functions

\[ f_v(z) = z - \frac{1 - \alpha}{(j + 1)^n(j + 1 - \alpha)[1 + [(j + 1)^m - 1] \lambda]^z} z^{j+1} \quad (v = 1, 2, 3). \] \quad (6.21)

**Proof.** From Theorem 10, we have

\[ f_1 * f_2(z) \in T_j(n, m, \lambda, \beta(j, n, m, \lambda, \alpha)), \]

where \( \beta \) is given by (6.2). Now, using Theorem 11, we get

\[ f_1 * f_2 * f_3(z) \in T_j(n, m, \lambda, \delta(j, n, m, \lambda, \alpha)), \]

where

\[ \delta(j, n, m, \lambda, \alpha) = 1 - \frac{j(1 - \alpha)(1 - \beta)}{(j + 1)^n(j + 1 - \alpha)(j + 1 - \beta)[1 + [(j + 1)^m - 1] \lambda] - (1 - \alpha)(1 - \beta)} \]

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\[
1 - \frac{j(1 - \alpha)^3}{(j + 1)^m(1 - \alpha)^3[1 + (j + 1)^m - 1\lambda]} - (1 - \alpha)^3.
\]

This completes the proof of Corollary 5.

**Theorem 12.** Let the functions \( f_v(z) \) \((v = 1, 2)\) defined by (4.1) be in the class \( T_j(n, m, \lambda, \alpha) \), then the function

\[
h(z) = z - \sum_{k=j+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k
\]

belongs to the class \( T_j(n, m, \lambda, \eta(j, n, m, \lambda, \alpha)) \), where

\[
\eta(j, n, m, \lambda, \alpha) = 1 - \frac{2j(1 - \alpha)^2}{(j + 1)^m(j + 1 - \alpha)^2[1 + (j + 1)^m - 1\lambda] - 2(1 - \alpha)^2}.
\]

The result is sharp for the functions \( f_v(z) \) \((v = 1, 2)\) defined by (6.14).

**Proof.** By virtue of Theorem 1, we obtain

\[
\sum_{k=j+1}^{\infty} \left[ \frac{k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha} \right]^2 a_{k,1}^2 \leq 1
\]

and

\[
\sum_{k=j+1}^{\infty} \left[ \frac{k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha} \right]^2 a_{k,2}^2 \leq 1.
\]

It follows from (6.24) and (6.25) that

\[
\sum_{k=j+1}^{\infty} \frac{1}{2} \left[ \frac{k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha} \right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1.
\]

Therefore, we need to find the largest \( \eta = \eta(j, n, m, \lambda, \alpha) \) such that

\[
k^n(k - \eta)[1 + (k^m - 1)\lambda] \leq \frac{1}{2} \left[ \frac{k^n(k - \alpha)[1 + (k^m - 1)\lambda]}{1 - \alpha} \right]^2 (k \geq j + 1),
\]

that is,

\[
\eta \leq 1 - \frac{2(k - 1)(1 - \alpha)^2}{(k - \alpha)^2 k^n[1 + (k^m - 1)\lambda] - 2(1 - \alpha)^2} (k \geq j + 1).
\]

Since

\[
B(k) = 1 - \frac{2(k - 1)(1 - \alpha)^2}{k^n(k - \alpha)^2[1 + (k^m - 1)\lambda] - 2(1 - \alpha)^2}.
\]

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is an increasing function of $k$ ($k \geq j + 1$), we readily have
\[ \eta \leq B(j + 1) = 1 - \frac{2j(1 - \alpha)^2}{(j + 1)^2[1 + [(j + 1)^m - 1][\lambda] - 2(1 - \alpha)^2}. \tag{6.30} \]
and Theorem 12 follows at once.

7. A family of integral operators

Theorem 13. Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$, and let $c$ be a real number such that $c > -1$. Then the function $F(z)$ defined by
\[ F(z) = \frac{c + 1}{z^n} \int_0^z t^{c-1} f(t) dt \quad (c > -1) \tag{7.1} \]
also belongs to the class $T_j(n, m, \lambda, \alpha)$.

Proof. From the representation (7.1) of $F(z)$, it follows that
\[ F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \]
where
\[ b_k = \left( \frac{c + 1}{c + k} \right) a_k. \]
Therefore, we have
\[ \sum_{k=j+1}^{\infty} k^n (k - \alpha)[1 + (k^m - 1)\lambda] b_k = \sum_{k=j+1}^{\infty} k^n (k - \alpha)[1 + (k^m - 1)\lambda] \left( \frac{c + 1}{c + k} \right) a_k \leq \]
\[ \sum_{k=j+1}^{\infty} k^n (k - \alpha)[1 + (k^m - 1)\lambda] a_k \leq 1 - \alpha, \]
since $f(z) \in T_j(n, m, \lambda, \alpha)$. Hence, by Theorem 1, $F(z) \in T_j(n, m, \lambda, \alpha)$.

Theorem 14. Let the function
\[ F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0, \ j \in \mathbb{N}) \]
be in the class $T_j(n, m, \lambda, \alpha)$, and let $c$ be a real number such that $c > -1$. Then the function $f(z)$ given by (7.1) is univalent in $|z| < R^*$, where
\[ R^* = \inf_k \left[ \frac{(k - \alpha) k^{n-1}[1 + (k^m - 1)\lambda](c + 1)}{(1 - \alpha)(c + k)} \right]^{\frac{1}{c+1}} \quad (k \geq j + 1). \tag{7.2} \]
The result is sharp.
**Proof.** From (7.1), we have

\[ f(z) = z^{1-c} (z^c F(z))' = z - \sum_{k=j+1}^{\infty} \left( \frac{c+k}{c+1} \right) a_k z^k. \]

In order to obtain the required result, it suffices to show that

\[ |f'(z) - 1| < 1 \text{ whenever } |z| < R^*, \]

where \( R^* \) is given by (7.2). Now

\[ |f'(z) - 1| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}. \]

Thus \( |f'(z) - 1| < 1 \) if

\[ \sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \] (7.3)

But Theorem 1 confirms that

\[ \sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)}{1-\alpha} \frac{1+(k^m-1)\lambda}{1+(k^m-1)\lambda} a_k \leq 1. \] (7.4)

Hence (7.3) will be satisfied if

\[ \frac{k(c+k)}{c+1} |z|^{k-1} < \frac{k^n(k-\alpha)}{1-\alpha} \frac{1+(k^m-1)\lambda}{1+(k^m-1)\lambda}, \]

that is, if

\[ |z| < \left( \frac{1+(k^m-1)\lambda}{1+(k^m-1)\lambda} \right)^{\frac{1}{k-n}} \left( \frac{k-\alpha}{1-\alpha} \right)^{\frac{1}{k}} (k \geq j+1). \] (7.5)

Therefore, the function \( f(z) \) given by (7.1) is univalent in \( |z| < R^* \). Sharpness of the result follows if we take

\[ f(z) = z - \frac{(1-\alpha)(c+k)}{k^n(k-\alpha)} \frac{1+(k^m-1)\lambda}{1+(k^m-1)\lambda} z^k \ (k \geq j+1). \] (7.6)

**References**


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