GENERALIZATION OF CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Dedicated to Professor Petru T. Mocanu on his 70th birthday

Abstract. The object of the present paper is to obtain coefficient estimates, some properties, distortion theorem and closure theorems for the classes \( R^*_n(\alpha) \) of analytic and univalent functions with negative coefficients, defined by using the \( n \)-th order Ruscheweyh derivative. We also obtain several interesting results for the modified Hadamard product of functions belonging to the classes \( R^*_n(\alpha) \). Further, we obtain radii of close-to-convexity, starlikeness and convexity and integral operators for the classes \( R^*_n(\alpha) \).

1. Introduction

Let \( A \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

which are analytic in the unit disc \( U = \{ z : |z| < 1 \} \). We denote by \( S \) the subclass of univalent functions \( f(z) \) in \( A \). The Hadamard product of two functions \( f(z) \in A \) and \( g(z) \in A \) will be denoted by \( f \ast g(z) \), that is, if \( f(z) \) is given by (1.1) and \( g(z) \) is given by

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k,
\]

then

\[
f \ast g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.
\]

Let

\[
D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}
\]

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for \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( z \in U \), where \( \mathbb{N} = \{1, 2, 3, \ldots\} \). This symbol \( D^n f(z) \) was named the \( n \)-th order Ruscheweyh derivative of \( f(z) \) by Al-Amiri [3]. We note that \( D^0 f(z) = f(z) \) and \( D^1 f(z) = z f'(z) \). By using the Hadamard product, Ruscheweyh [5] observed that if

\[
D^\beta f(z) = \frac{z}{(1 - z)^{\beta + 1}} \ast f(z) \quad (\beta \geq -1)
\]

then (1.4) is equivalent to (1.5) when \( \beta = n \in \mathbb{N}_0 \).

It is easy to see that

\[
D^n f(z) = k + \sum_{k=2}^{\infty} \delta(n, k) a_k z^k,
\]

where

\[
\delta(n, k) = \binom{n + k - 1}{n}.
\]

Note that

\[
z(D^n f(z))' = (n + 1)D^{n+1} f(z) - nD^n f(z) \quad (\text{cf. [5]}).
\]

Let \( R_n(\alpha) \) denote the classes of functions \( f(z) \in A \) which satisfy the condition

\[
\text{Re} \left\{ \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \alpha, \quad (z \in U)
\]

for some \( \alpha \) \((0 \leq \alpha < 1) \) and \( n \in \mathbb{N}_0 \). The class \( R_n(\alpha) \) was studied by Ahuja [1,2].

From (1.8) and (1.9) it follows that a function \( f(z) \) in \( A \) belongs to \( R_n(\alpha) \) if and only if

\[
\text{Re} \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \frac{n + \alpha}{n + 1}, \quad (z \in U).
\]

Let \( T \) denote the subclass of \( S \) consisting of functions \( f(z) \) of the form

\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0).
\]

In the present paper we introduce the following classes \( R_n^*(\alpha) \) by using the \( n \)-th order Ruscheweyh derivative of \( f(z) \), defined as follows:

**Definition.** We say that \( f(z) \) is in the class \( R_n^*(\alpha) \) \((0 \leq \alpha < 1, \ n \in \mathbb{N}_0) \), if \( f(z) \) defined by (1.11) satisfies the condition (1.10).

We note that \( R_n^*(0) = R_n^* \) was studied by Owa [4] and \( R_n^*(\alpha) = T^*(\alpha) \) (the class of starlike functions of order \( \alpha \)) and \( R_1^*(\alpha) = C(\alpha) \) (the class of convex functions
of order \( \alpha \), were studied by Silverman [7]. Hence \( R^*_n(\alpha) \) is a subclass of \( T^*(\alpha) \subset S \). Further, we can show that \( R^*_{n+1}(\alpha) \subset R^*_n(\alpha) \) for every \( n \in \mathbb{N}_0 \).

2. Coefficient Estimates

**Theorem 1.** Let the function \( f(z) \) be defined by (1.11). Then \( f(z) \) is in the class \( R^*_n(\alpha) \) if and only if

\[
\sum_{k=2}^{\infty} (k - \alpha)\delta(n,k)a_k \leq 1 - \alpha. \tag{2.1}
\]

The result is sharp.

**Proof.** Assume that the inequality (2.1) holds and let \( |z| = 1 \). Then we get

\[
\left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| = \left| \frac{-\sum_{k=2}^{\infty} (\delta(n+1,k) - \delta(n,k))a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} \delta(n,k)a_k z^{k-1}} \right| \leq \frac{1 - \sum_{k=2}^{\infty} \delta(n,k)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} \delta(n,k)a_k} \leq \frac{1 - \alpha}{n + 1}.
\]

This shows that the values of \( \frac{D^{n+1}f(z)}{D^n f(z)} \) lies in a circle centered at \( w = 1 \) whose radius is \( \frac{1 - \alpha}{n + 1} \). Hence \( f(z) \) satisfies the condition (1.10) hence further, \( f(z) \in R^*_n(\alpha) \).

For the converse, assume that the function \( f(z) \) defined by (1.11) belongs to the class \( R^*_n(\alpha) \). Then we have

\[
\text{Re} \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} = \text{Re} \left\{ \frac{1 - \sum_{k=2}^{\infty} \delta(n+1,k)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} \delta(n,k)a_k z^{k-1}} \right\} \geq \frac{n + \alpha}{n + 1} \tag{2.2}
\]

for \( 0 \leq \alpha < 1 \) and \( z \in U \). Choose values of \( z \) on the real axis so that \( \frac{D^{n+1}f(z)}{D^n f(z)} \) is real. Upon clearing the denominator in (2.2) and letting \( z \to 1^- \) through real values, we get

\[
(n + 1) \left( 1 - \sum_{k=2}^{\infty} \delta(n+1,k)a_k \right) \geq (n + \alpha) \left( 1 - \sum_{k=2}^{\infty} \delta(n,k)a_k \right) \tag{2.3}
\]
which gives (2.1). Finally the function \( f(z) \) given by

\[
f(z) = z - \frac{1 - \alpha}{(k - \alpha)\delta(n, k)} z^k \quad (k \geq 2)
\]  

is an extremal function for the theorem.

**Corollary 1.** Let the function \( f(z) \) defined by (1.11) be in the class \( R_n^*(\alpha) \). Then

\[
a_k \leq \frac{1 - \alpha}{(k - \alpha)\delta(n, k)} \quad (k \geq 2).
\]  

The equality in (2.5) is attained for the function \( f(z) \) given by (2.4).

3. Some properties of the class \( R_n^*(\alpha) \)

**Theorem 2.** Let \( 0 \leq \alpha_1 \leq \alpha_2 < 1 \) and \( n \in \mathbb{N}_0 \). Then we have

\[
R_n^*(\alpha_1) \supseteq R_n^*(\alpha_2).
\]  

**Proof.** Let the function \( f(z) \) defined by (1.11) be in the class \( R_n^*(\alpha_2) \) and \( \alpha_1 = \alpha_2 - \varepsilon \). Then, by Theorem 1, we have

\[
\sum_{k=2}^{\infty} (k - \alpha_2)\delta(n, k)a_k \leq 1 - \alpha_2
\]

and

\[
\sum_{k=2}^{\infty} \delta(n, k)a_k \leq \frac{1 - \alpha_2}{2 - \alpha_2} < 1.
\]  

Consequently

\[
\sum_{k=2}^{\infty} (k - \alpha_1)\delta(n, k)a_k = \sum_{k=2}^{\infty} (k - \alpha_2)\delta(n, k)a_k + \varepsilon \sum_{k=2}^{\infty} \delta(n, k)a_k \leq 1 - \alpha_1.
\]  

This completes the proof of Theorem 2 with the aid of Theorem 1.

**Theorem 3.** \( R_{n+1}^*(\alpha) \subseteq R_n^*(\alpha) \) for \( 0 \leq \alpha < 1 \) and \( n \in \mathbb{N}_0 \).

**Proof.** Let the function \( f(z) \) defined by (1.11) be in the class \( R_{n+1}^*(\alpha) \); then

\[
\sum_{k=2}^{\infty} (k - \alpha)\delta(n + 1, k)a_k \leq 1 - \alpha
\]

and since

\[
\delta(n, k) \leq \delta(n + 1, k) \text{ for } k \geq 2,
\]
we have
\[ \sum_{k=2}^{\infty} (k - \alpha) \delta(n, k) a_k \leq \sum_{k=2}^{\infty} (k - \alpha) \delta(n + 1, k) a_k \leq 1 - \alpha. \] (3.6)

The result follows from Theorem 1.

4. Distortion theorem

Theorem 4. Let the function \( f(z) \) defined by (1.11) be in the class \( R_n^*(\alpha) \).

Then we have for \( |z| = r < 1 \)
\[ r - \frac{1 - \alpha}{(2 - \alpha)(n + 1)} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{(2 - \alpha)(n + 1)} r^2 \] (4.1)
and
\[ 1 - \frac{2(1 - \alpha)}{(2 - \alpha)(n + 1)} r \leq |f'(z)| \leq 1 + \frac{2(1 - \alpha)}{(2 - \alpha)(n + 1)} r. \] (4.2)

The result is sharp.

Proof. In view of Theorem 1, we have
\[ \sum_{k=2}^{\infty} a_k \leq \frac{1 - \alpha}{(2 - \alpha)(n + 1)}. \] (4.3)

Consequently, we have
\[ |f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - \frac{1 - \alpha}{(2 - \alpha)(n + 1)} r^2 \] (4.4)
and
\[ |f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k \leq r + \frac{1 - \alpha}{(2 - \alpha)(n + 1)} r^2 \] (4.5)
which prove the assertion (4.1).

From (4.3) and Theorem 1, it follows also that
\[ \sum_{k=2}^{\infty} ka_k \leq \frac{1 - \alpha}{n + 1} + \alpha \sum_{k=2}^{\infty} a_k \leq \frac{2(1 - \alpha)}{(2 - \alpha)(n + 1)}. \] (4.6)

Consequently, we have
\[ |f'(z)| \geq 1 - r \sum_{k=2}^{\infty} ka_k \geq 1 - \frac{2(1 - \alpha)}{(2 - \alpha)(n + 1)} r \] (4.7)
and
\[ |f'(z)| \leq 1 + r \sum_{k=2}^{\infty} ka_k \leq 1 + \frac{2(1 - \alpha)}{(2 - \alpha)(n + 1)} r, \] (4.8)
which prove the assertion (4.2). This completes the proof of Theorem 4.
The bounds in (4.1) and (4.2) are attained for the function \( f(z) \) given by
\[
f(z) = z - \frac{1 - \alpha}{(2 - \alpha)(n + 1)} z^2 \quad (z = \pm r).
\]

**Corollary 2.** Let the function \( f(z) \) defined by (1.11) be in the class \( R^*_n(\alpha) \). Then the unit disc \( \mathbb{U} \) is mapped onto a domain that contains the disc
\[
|w| < \frac{(2 - \alpha)(n + 1) - (1 - \alpha)}{(2 - \alpha)(n + 1)}
\]
(4.10)
The result is sharp with extremal function \( f(z) \) given by (4.9).

5. **Closure theorems**

Let the functions \( f_i(z) \) be defined, for \( i = 1, 2, \ldots, m \), by
\[
f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (a_{k,i} \geq 0, \ k \geq 2)
\]
(5.1)
for \( z \in \mathbb{U} \).

We shall prove the following results for the closure of functions in the classes \( R^*_n(\alpha) \).

**Theorem 5.** Let the functions \( f_i(z) \) defined by (5.1) be in the class \( R^*_n(\alpha) \) for every \( i = 1, 2, \ldots, m \). Then the function \( h(z) \) defined by
\[
h(z) = \sum_{i=1}^{m} c_i f_i(z) \quad (c_i \geq 0)
\]
(5.2)
is also in the class \( R^*_n(\alpha) \), where
\[
\sum_{i=1}^{m} c_i = 1.
\]
(5.3)

**Proof.** According to the definition of \( h(z) \), we can write
\[
h(z) = z - \sum_{k=2}^{\infty} \left( \sum_{i=1}^{m} c_i a_{k,i} \right) z^k.
\]
(5.4)
Further, since \( f_i(z) \) are in \( R^*_n(\alpha) \) for every \( i = 1, 2, \ldots, m \), we get
\[
\sum_{k=2}^{\infty} (k - \alpha) \delta(n, k) a_{k,i} \leq 1 - \alpha
\]
(5.5)
for every \( i = 1, 2, \ldots, m \). Hence we can see that
\[
\sum_{k=2}^{\infty} (k - \alpha) \delta(n, k) \left( \sum_{i=1}^{m} c_i a_{k,i} \right) = \sum_{i=1}^{m} c_i \left( \sum_{k=2}^{\infty} (k - \alpha) \delta(n, k) a_{k,i} \right) \leq
\]
with the aid of (5.5). This proves that the function \( h(z) \) is in the class \( R_n^*(\alpha) \) by means of Theorem 1. Thus we have the theorem.

**Theorem 6.** The class \( R_n^*(\alpha) \) is closed under convex linear combinations.

**Proof.** Let the functions \( f_i(z) \) \((i = 1, 2)\) defined by (5.1) be in the class \( R_n^*(\alpha) \). Then it is sufficient to prove that the function

\[
h(z) = \lambda f_1(z) + (1 - \lambda) f_2(z) \quad (0 \leq \lambda \leq 1)
\]

is in the class \( R_n^*(\alpha) \). Since, for \( 0 \leq \lambda \leq 1 \),

\[
h(z) = z - \sum_{k=2}^{\infty} \{\lambda a_{k,1} + (1 - \lambda) a_{k,2}\} z^k,
\]

we readily have

\[
\sum_{k=2}^{\infty} (k - \alpha) \delta(n, k) \{\lambda a_{k,1} + (1 - \lambda) a_{k,2}\} \leq 1 - \alpha,
\]

by means of Theorem 1, which implies that \( h(z) \in R_n^*(\alpha) \).

**Theorem 7.** Let

\[
f_1(z) = z
\]

and

\[
f_k(z) = z - \frac{1 - \alpha}{(k - \alpha) \delta(n, k)} z^k \quad (k \geq 2)
\]

for \( 0 \leq \alpha < 1 \) and \( n \in \mathbb{N}_0 \). Then \( f(z) \) is in the class \( R_n^*(\alpha) \) if and only if can be expressed in the form

\[
f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)
\]

where \( \lambda_k \geq 0 \) and

\[
\sum_{k=1}^{\infty} \lambda_k = 1.
\]

**Proof.** Assume that

\[
f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1 - \alpha}{(k - \alpha) \delta(n, k)} \lambda_k z^k.
\]
Then we have
\[
\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) \cdot 1 - \frac{1}{1 - \alpha} \lambda_k = \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1.
\] (5.15)

So by Theorem 1, \( f(z) \in R^*_n(\alpha) \).

Conversely, assume that the function \( f(z) \) defined by (1.11) belongs to the class \( R^*_n(\alpha) \). Again, with the aid of Theorem 1, we get
\[
a_k \leq \frac{1 - \alpha}{(k-\alpha)\delta(n,k)} (k \geq 2).
\] (5.16)

Setting
\[
\lambda_k = \frac{(k-\alpha)\delta(n,k)}{1 - \alpha} a_k \quad (k \geq 2),
\] (5.17)
and
\[
\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.
\] (5.18)

Hence, we can see that \( f(z) \) can be expressed in the form (5.12). This completes the proof of Theorem 7.

**Corollary 3.** The extreme points of the class \( R^*_n(\alpha) \) are the functions \( f_1(z) \) and \( f_k(z) \) \((k \geq 2)\) given by Theorem 7.

### 6. Modified Hadamard product

Let the functions \( f_i(z) \) \((i = 1, 2)\) be defined (5.1). The modified Hadamard product of \( f_1(z) \) and \( f_2(z) \) is defined by
\[
f_1 \ast f_2(z) = z - \sum_{k=2}^{\infty} a_{1,k} a_{2,k} z^k.
\] (6.1)

**Theorem 8.** Let the functions \( f_i(z) \) \((i = 1, 2)\) be defined by (5.1) be in the class \( R^*_n(\alpha) \). Then \( f_1 \ast f_2(z) \in R^*_n(\beta(n,\alpha)) \), where
\[
\beta(n,\alpha) = \frac{(n+1) - 2 \left(\frac{1 - \alpha}{2 - \alpha}\right)^2}{(n+1) - \left(\frac{1 - \alpha}{2 - \alpha}\right)^2}.
\] (6.2)

The result is sharp.
Proof. Employing the technique used earlier by Schild and Silverman [4], we need to find the largest $\beta = \beta(n, \alpha)$ such that

$$\sum_{k=2}^{\infty} \frac{(k - \beta)\delta(n, k)}{1 - \beta} a_{k,1} a_{k,2} \leq 1.$$  \hspace{1cm} (6.3)

Since

$$\sum_{k=2}^{\infty} \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} a_{k,1} \leq 1$$  \hspace{1cm} (6.4)

and

$$\sum_{k=2}^{\infty} \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} a_{k,2} \leq 1,$$  \hspace{1cm} (6.5)

by the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} \sqrt{a_{k,1} a_{k,2}} \leq 1.$$  \hspace{1cm} (6.6)

Thus it is sufficient to show that

$$\frac{(k - \beta)\delta(n, k)}{1 - \beta} a_{k,1} a_{k,2} \leq \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} \sqrt{a_{k,1} a_{k,2}} \quad (k \geq 2),$$  \hspace{1cm} (6.7)

that is, that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{(k - \alpha)(1 - \beta)}{(k - \beta)(1 - \alpha)} \quad (k \geq 2).$$  \hspace{1cm} (6.8)

Note that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{1 - \alpha}{(k - \beta)(1 - \alpha)} \quad (k \geq 2).$$  \hspace{1cm} (6.9)

Consequently, we need only to prove that

$$\frac{1 - \alpha}{(k - \alpha)\delta(n, k)} \leq \frac{(k - \alpha)(1 - \beta)}{(k - \beta)(1 - \alpha)} \quad (k \geq 2),$$  \hspace{1cm} (6.10)

or, equivalently, that

$$\beta \leq \frac{\delta(n, k) - k \left(\frac{1 - \alpha}{k - \alpha}\right)^2}{\delta(n, k) - \left(\frac{1 - \alpha}{k - \alpha}\right)^2} \quad (k \geq 2).$$  \hspace{1cm} (6.11)

Since

$$A(k) = \frac{\delta(n, k) - k \left(\frac{1 - \alpha}{k - \alpha}\right)^2}{\delta(n, k) - \left(\frac{1 - \alpha}{k - \alpha}\right)^2} \quad (6.12)$$
is an increasing function of \( k \) \((k \geq 2)\), letting \( k = 2 \) in (6.12), we obtain

\[
\beta \leq A(2) = \frac{(n + 1) - 2 \left( \frac{1 - \alpha}{2 - \alpha} \right)^2}{(n + 1) - \left( \frac{1 - \alpha}{2 - \alpha} \right)^2}, \tag{6.13}
\]

which completes the proof of the theorem. Finally, by taking the functions \( f_i(z) \) given by

\[
f_i(z) = z - \frac{1 - \alpha}{(2 - \alpha)(n + 1)} z^2 \quad (i = 1, 2), \tag{6.14}
\]

we can see that the result is sharp.

**Corollary 4.** For \( f_1(z) \) and \( f_2(z) \) as in Theorem 8, we have

\[
h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1}a_{k,2}} z^k \tag{6.15}
\]

belongs to the class \( R^*_n(\alpha) \).

The result follows from the inequality (6.6). It is sharp for the same functions \( f_i(z) \) \((i = 1, 2)\) as in Theorem 8.

**Theorem 9.** Let \( f_1(z) \in R^*_n(\alpha) \) and \( f_2(z) \in R^*_n(\beta) \), then \( f_1 \ast f_2(z) \in R^*_n(\gamma(n, \alpha, \beta)) \), where

\[
\gamma(n, \alpha, \beta) = \frac{(n + 1) - 2 \left( \frac{1 - \alpha}{2 - \alpha} \right) \left( \frac{1 - \beta}{2 - \beta} \right)}{(n + 1) - \left( \frac{1 - \alpha}{2 - \alpha} \right) \left( \frac{1 - \beta}{2 - \beta} \right)}. \tag{6.16}
\]

The result is sharp for the functions

\[
f_1(z) = z - \frac{1 - \alpha}{(2 - \alpha)(n + 1)} z^2 \tag{6.17}
\]

and

\[
f_2(z) = z - \frac{1 - \beta}{(2 - \beta)(n + 1)} z^2. \tag{6.18}
\]

**Proof.** Proceeding as in the proof of Theorem 8, we get

\[
\gamma \leq B(k) = \frac{\delta(n, k) - k \left( \frac{1 - \alpha}{k - \alpha} \right) \left( \frac{1 - \beta}{k - \beta} \right)}{\delta(n, k) - \left( \frac{1 - \alpha}{k - \alpha} \right) \left( \frac{1 - \beta}{k - \beta} \right)}. \tag{6.19}
\]
Since the function $B(k)$ is an increasing function of $k$ ($k \geq 2$), setting $k = 2$ in (6.19), we obtain

$$\gamma \leq B(2) = \frac{(n + 1) - 2 \left(\frac{1 - \alpha}{2 - \alpha}\right) \left(\frac{1 - \beta}{2 - \beta}\right)}{(n + 1) - \left(\frac{1 - \alpha}{2 - \alpha}\right) \left(\frac{1 - \beta}{2 - \beta}\right)}.$$  \hspace{1cm} (6.20)

This completes the proof of Theorem 9.

**Corollary 5.** Let the functions $f_i(z)$ ($i = 1, 2, 3$) defined by (5.1) be in the class $R^*_n(\alpha)$, then $f_1 * f_2 * f_3(z) \in R^*_n(\zeta(n, \alpha))$, where

$$\zeta(n, \alpha) = \frac{(n + 1)^2 - 2 \left(\frac{1 - \alpha}{2 - \alpha}\right)^3}{(n + 1)^2 - \left(\frac{1 - \alpha}{2 - \alpha}\right)^3}.$$ \hspace{1cm} (6.21)

The result is best possible for the functions

$$f_i(z) = z - \frac{1 - \alpha}{(2 - \alpha)(n + 1)} z^2 \quad (i = 1, 2, 3).$$ \hspace{1cm} (6.22)

**Proof.** From Theorem 8, we have $f_1 * f_2(z) \in R^*_n(\beta)$, where $\beta$ is given by (6.2). We use now Theorem 9, we get $f_1 * f_2 * f_3(z) \in R^*_n(\zeta(n, \alpha))$, where

$$\zeta(n, \alpha) = \frac{(n + 1) - 2 \left(\frac{1 - \alpha}{2 - \alpha}\right) \left(\frac{1 - \beta}{2 - \beta}\right)}{(n + 1) - \left(\frac{1 - \alpha}{2 - \alpha}\right) \left(\frac{1 - \beta}{2 - \beta}\right)} = \frac{(n + 1)^2 - 2 \left(\frac{1 - \alpha}{2 - \alpha}\right)^3}{(n + 1)^2 - \left(\frac{1 - \alpha}{2 - \alpha}\right)^3}.$$  \hspace{1cm} (6.24)

This completes the proof of Corollary 5.

**Theorem 10.** Let the functions $f_i(z)$ ($i = 1, 2$) defined by (5.1) be in the class $R^*_n(\alpha)$. Then the function

$$h(z) = z - \sum_{k=2}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k$$ \hspace{1cm} (6.23)

belongs to the class $R^*_n(\varphi(n, \alpha))$, where

$$\varphi(n, \alpha) = \frac{(n + 1) - \left(\frac{2(1 - \alpha)}{2 - \alpha}\right)^2}{(n + 1) - 2 \left(\frac{1 - \alpha}{2 - \alpha}\right)^2}.$$ \hspace{1cm} (6.24)

The result is sharp for the functions $f_i(z)$ ($i = 1, 2$) defined by (6.14).
Proof. By virtue of Theorem 1, we obtain
\[
\sum_{k=2}^{\infty} \left( \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} \right)^2 a_{k,1}^2 \leq \left( \sum_{k=2}^{\infty} \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} a_{k,1} \right)^2 \leq 1
\]
and
\[
\sum_{k=2}^{\infty} \left( \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} \right)^2 a_{k,2}^2 \leq \left( \sum_{k=2}^{\infty} \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} a_{k,2} \right)^2 \leq 1.
\]
It follows from (6.25) and (6.26) that
\[
\sum_{k=2}^{\infty} \left( \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} \right)^2 a_{k,1}^2 \leq \sum_{k=2}^{\infty} \left( \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} \right)^2 a_{k,2}^2 \leq 1.
\]
Therefore, we need to find the largest \( \varphi = \varphi(n, \alpha) \) such that
\[
\frac{(k - \varphi)\delta(n, k)}{1 - \varphi} \leq \frac{1}{2} \left( \frac{(k - \alpha)\delta(n, k)}{1 - \alpha} \right)^2 (k \geq 2),
\]
that is
\[
\varphi \leq \frac{\delta(n, k) - 2k \left( \frac{1 - \alpha}{k - \alpha} \right)^2}{\delta(n, k) - 2 \left( \frac{1 - \alpha}{k - \alpha} \right)^2} (k \geq 2).
\]
Since
\[
D(k) = \frac{\delta(n, k) - 2k \left( \frac{1 - \alpha}{k - \alpha} \right)^2}{\delta(n, k) - 2 \left( \frac{1 - \alpha}{k - \alpha} \right)^2}
\]
is an increasing function of \( k \) \((k \geq 2)\), we readily have
\[
\varphi \leq D(2) = \frac{(n + 1) - \frac{2(1 - \alpha)}{2 - \alpha}}{(n + 1) - 2 \left( \frac{1 - \alpha}{2 - \alpha} \right)^2},
\]
and Theorem 10 follows at once.

Theorem 11. Let \( f_1(z) \in R^*_{n_1}(\alpha) \), and \( f_2(z) \in R^*_{n_2}(\alpha) \). Then the modified Hadamard product \( f_1 \ast f_2(z) \in R^*_{n_1}(\alpha) \cap R^*_{n_2}(\alpha) \).

Proof. Since \( f_2(z) \in R^*_{n_2}(\alpha) \), we have from (4.3) that
\[
a_{k,2} \leq \frac{1 - \alpha}{(2 - \alpha)(n_2 + 1)}. \tag{6.32}
\]
From Theorem 1, since \( f_1(z) \in R^*_{n_1}(\alpha) \), we have
\[
\sum_{k=2}^{\infty} \frac{(k - \alpha)\delta(n_1, k)}{1 - \alpha} a_{k,1} \leq 1. \tag{6.33}
\]
Now, from (6.32) and (6.33), we have
\[
\sum_{k=2}^{\infty} \frac{(k - \alpha)\delta(n_1, k)}{1 - \alpha} a_{k,1} a_{k,2} \leq \frac{1 - \alpha}{(2 - \alpha)(n_2 + 1)} \sum_{k=2}^{\infty} \frac{(k - \alpha)\delta(n_1, k)}{1 - \alpha} a_{k,1} \leq \frac{1 - \alpha}{(2 - \alpha)(n_2 + 1)} \leq 1.
\]
Hence \(f_1 * f_2(z) \in R_{n_1}^*(\alpha)\). Interchanging \(n_1\) and \(n_2\) by each other in the above, we get \(f_1 * f_2(z) \in R_{n_2}^*(\alpha)\). Hence the theorem.

7. Radii of close-to-convexity, starlikeness and convexity

**Theorem 12.** Let the function \(f(z)\) defined by (1.11) be in the class \(R_{n_1}^*(\alpha)\), then \(f(z)\) is close-to-convex of order \(\rho\) (0 \(\leq \rho < 1\)) in \(|z| < r_1(n, \alpha, \rho)\), where

\[
r_1(n, \alpha, \rho) = \inf_k \left[ \frac{(1 - \rho)(k - \alpha)\delta(n, k)}{k(1 - \alpha)} \right]^{\frac{1}{k}} (k \geq 2). \tag{7.1}
\]

The result is sharp, with the extremal function \(f(z)\) given by (2.4).

**Proof.** We must show that \(|f'(z) - 1| \leq 1 - \rho\) for \(|z| < r_1(n, \alpha, \rho)\). We have

\[
|f'(z) - 1| \leq \sum_{k=2}^{\infty} ka_k |z|^{k-1}.
\]

Thus \(|f'(z) - 1| \leq 1 - \rho\) if

\[
\sum_{k=2}^{\infty} \left( \frac{k}{1 - \rho} \right) a_k |z|^{k-1} \leq 1. \tag{7.2}
\]

Hence, by Theorem 1, (7.2) will be true if

\[
\frac{k|z|^{k-1}}{1 - \rho} \leq \frac{(k - \alpha)\delta(n, k)}{1 - \alpha}
\]
or if

\[
|z| \leq \left[ \frac{(1 - \rho)(k - \alpha)\delta(n, k)}{k(1 - \alpha)} \right]^{\frac{1}{1-\rho}} (k \geq 2). \tag{7.3}
\]

The theorem follows easily from (7.3).

**Theorem 13.** Let the function \(f(z)\) defined by (1.11) be in the class \(R_{n_1}^*(\alpha)\), then \(f(z)\) is starlike of order \(\rho\) (0 \(\leq \rho < 1\)) in \(|z| < r_2(n, \alpha, \rho)\), where

\[
r_2(n, \alpha, \rho) = \inf_k \left[ \frac{(1 - \rho)(k - \alpha)\delta(n, k)}{(k - \rho)(1 - \alpha)} \right]^{\frac{1}{1-\rho}} (k \geq 2). \tag{7.4}
\]

The result is sharp, with the extremal function \(f(z)\) given by (2.4).
**Proof.** It is sufficient to show that 
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \text{ for } |z| < r_2(n, \alpha, \rho). \]

We have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \sum_{k=2}^{\infty} \frac{(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k|z|^{k-1}}.
\]

Thus
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \text{ if } \sum_{k=2}^{\infty} \frac{(k-\rho)a_k|z|^{k-1}}{1 - \rho} \leq 1. \quad (7.5)
\]

Hence, by Theorem 1, (7.5) will be true if
\[
\frac{(k-\rho)|z|^{k-1}}{1 - \rho} \leq \frac{(k-\alpha)\delta(n,k)}{1 - \alpha}
\]
or if
\[
|z| \leq \left[ \frac{(1-\rho)(k-\alpha)\delta(n,k)}{(k-\rho)(1-\alpha)} \right]^{1/k} \quad (k \geq 2). \quad (7.6)
\]
The theorem follows easily from (7.6).

**Corollary 6.** Let the function \( f(z) \) defined by (1.11) be in the class \( R^*_n(\alpha) \), then \( f(z) \) is convex of order \( \rho \) \( (0 \leq \rho < 1) \) in \( |z| < \rho_3(n, \alpha, \rho) \), where
\[
\rho_3(n, \alpha, \rho) = \inf_k \left[ \frac{(1-\rho)(k-\alpha)\delta(n,k)}{k(k-\rho)(1-\alpha)} \right]^{1/k} \quad (k \geq 2). \quad (7.7)
\]
The result is sharp, with the extremal function \( f(z) \) given by (2.4).

8. **Integral operators**

**Theorem 14.** Let the function \( f(z) \) defined by (1.11) be in the class \( R^*_n(\alpha) \) and let the function \( F(z) \) be defined by
\[
F(z) = \frac{c+1}{c} \int_0^z t^{c-1} f(t) dt. \quad (8.1)
\]

Then

(i) for every \( c \), \( c > -1 \), \( F(z) \in R^*_n(\alpha) \)

and

(ii) for every \( c \), \( -1 < c \leq n \), \( F(z) \in R^*_{n+1}(\alpha) \).
Proof. (i) From the representation of \( F(z) \), it follows that
\[
F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{8.2}
\]
where
\[
b_k = \left( \frac{c + 1}{c + k} \right) a_k. \tag{8.3}
\]
Therefore,
\[
\sum_{k=2}^{\infty} (k - \alpha) \delta(n, k) b_k = \sum_{k=2}^{\infty} (k - \alpha) \delta(n, k) \left( \frac{c + 1}{c + k} \right) a_k \leq \sum_{k=2}^{\infty} (k - \alpha) \delta(n, k) a_k \leq 1 - \alpha,
\]
since \( f(z) \in R_n^*(\alpha) \). Hence, by Theorem 1, \( F(z) \in R_n^*(\alpha) \).

(ii) In view of Theorem 1 it is sufficient to show that
\[
\sum_{k=2}^{\infty} (k - \alpha) \delta(n + 1, k) \left( \frac{c + 1}{c + k} \right) a_k \leq 1 - \alpha. \tag{8.4}
\]
Since
\[
\delta(n, k) - \delta(n + 1, k) \left( \frac{c + 1}{c + k} \right) \geq 0 \text{ if } -1 < c \leq n (k = 2, 3, \ldots)
\]
the result follows from Theorem 1.

Putting \( c = 0 \) in Theorem 14, we get

**Corollary 7.** Let the function \( f(z) \) defined by (1.6) be in the class \( R_n^*(\alpha) \) and let the function \( F(z) \) be defined by
\[
F(z) = \int_0^z \frac{f(t)}{t} \, dt. \tag{8.5}
\]
Then \( F(z) \in R_{n+1}^*(\alpha) \).

**Theorem 15.** Let the function \( F(z) = z - \sum_{k=2}^{\infty} a_k z^k \) \((a_k \geq 0)\) be in the class \( R_n^*(\alpha) \), and let \( c \) be a real number such that \( c > -1 \). Then the function \( f(z) \) defined by (8.1) is univalent in \(|z| < r^*\), where
\[
r^* = \inf_k \left[ \frac{(c+1)(k-\alpha)\delta(n,k)}{k(c+k)(1-\alpha)} \right]^{\frac{1}{k-1}}, \quad (k \geq 2). \tag{8.6}
\]
The result is sharp.

**Proof.** From (8.1), we have
\[
f(z) = \frac{z^{1-c} (z^c F(z))'}{c + 1} \quad (c > -1) = z - \sum_{k=2}^{\infty} \left( \frac{c + k}{c + 1} \right) a_k z^k. \tag{8.7}
\]
In order to obtain the required result it suffices to show that

\[ |f'(z) - 1| < 1 \text{ in } |z| < r^*. \]

Now

\[ |f'(z) - 1| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k|z|^{k-1}. \]

Thus \( |f'(z) - 1| < 1 \), if

\[ \sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k|z|^{k-1} < 1. \quad (8.8) \]

But Theorem 1 confirms that

\[ \sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_k \leq 1. \quad (8.9) \]

Hence (8.8) will be satisfied if

\[ \frac{k(c+k)|z|^{k-1}}{c+1} < \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \quad (k \geq 2) \]

or if

\[ |z| < \left[ \frac{(c+1)(k-\alpha)\delta(n,k)}{k(c+k)(1-\alpha)} \right]^\frac{1}{k-1} \quad (k \geq 2). \quad (8.10) \]

Therefore \( f(z) \) is univalent in \( |z| < r^* \). Sharpness follows if we take

\[ f(z) = z - \frac{(1-\alpha)(c+k)}{(k-\alpha)\delta(n,k)(c+1)} z^k \quad (k \geq 2). \quad (8.11) \]

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References

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