EXPONENTIAL STABILITY OF EVOLUTION OPERATORS

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Abstract. The aim of this paper is to give some sufficient, respectively necessary and sufficient conditions, for the exponential stability of evolution operators in infinite-dimensional spaces. The obtained results are like those, of Datko-type, for evolutionary processes which are linear operators-valued.

1. Introduction

Let $X$ be a Banach space and let $(X_t)_{t \geq 0}$ be a family of parts of $X$.

Definition 1. The family of applications $\Phi(t, t_0) : X_{t_0} \rightarrow X_t$, $t \geq t_0 \geq 0$, will be called an evolution operator in $X$, if the following conditions are satisfied:

i) $\Phi(t, t)x = x$, for all $t \geq 0$ and $x \in X_t$.

ii) $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$, for all $t \geq s \geq t_0 \geq 0$.

iii) $\Phi(\cdot, s)x : [s, \infty) \rightarrow X$ is continuous, for all $s \geq 0$ and $x \in X_s$.

iv) There is a nondecreasing function $p(\cdot) : \mathbb{R}_+ \rightarrow (0, \infty)$, such that

$$\|\Phi(t, s)x\| \leq p(t - s)\|x\|, \text{ for all } t \geq s \geq 0 \text{ and } x \in X_s.$$ 

Remark 1. Condition iv) can be replaced by

v) There are $M, \omega > 0$ such that

$$\|\Phi(t, s)x\| \leq Me^{\omega(t - s)}\|x\|, \text{ for all } t \geq s \geq 0 \text{ and } x \in X_s.$$ 

Proof. Let iv) be satisfied and let $t \geq s \geq 0$ and $x \in X_s$. Then there are $n \in \mathbb{N}$ and $r \in [0, 1)$ such that $t - s = n + r$. We have

$$\|\Phi(t, s)x\| \leq p(t - s - n)\|\Phi(s + n, s)x\| \leq p(1)^{n+1}\|x\|.$$ 

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Let $\omega > \max\{0, \ln p(1)\}$. Then

$$\|\Phi(t, s)x\| \leq p(1)e^{\omega n}\|x\| \leq p(1)e^{\omega (t-s)}\|x\|.$$ 

The converse is obviously. □

In the sequel we will denote by $M$ and $\omega$ those constants which satisfy condition v).

**Definition 2.** The evolution operator $\Phi(\cdot, \cdot)$ will be called exponentially stable, if there are $\nu > 0$ and a function $N(\cdot) : \mathbb{R}_+ \to (0, \infty)$ such that

$$\|\Phi(t, t_0)x\| \leq N(t_0)e^{-\nu (t-t_0)}\|x\|,$$

for all $t \geq t_0 \geq 0$ and $x \in X_{t_0}$.

**Remark 2.** Let $\Phi(\cdot, \cdot)$ be an evolution operator. The following assertions are equivalent:

1. $\Phi(\cdot, \cdot)$ is exponentially stable.
2. There are $\nu > 0$ and $N(\cdot) : \mathbb{R}_+ \to (0, \infty)$ such that

$$\|\Phi(t, t_0)x\| \leq N(s)e^{-\nu (t-s)}\|\Phi(s, t_0)x\|,$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X_{t_0}$.

**Definition 3.** The evolution operator $\Phi(\cdot, \cdot)$ will be called uniformly exponentially stable, if there are $N, \nu > 0$ such that

$$\|\Phi(t, t_0)x\| \leq Ne^{-\nu (t-t_0)}\|x\|,$$

for all $t \geq t_0 \geq 0$ and $x \in X_{t_0}$.

**Remark 3.** The evolution operator $\Phi(\cdot, \cdot)$ is uniformly exponentially stable if and only if there are $N, \nu > 0$ such that

$$\|\Phi(t, t_0)x\| \leq Ne^{-\nu (t-s)}\|\Phi(s, t_0)x\|,$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X_{t_0}$.

**Lemma.** Let $\Phi(\cdot, \cdot)$ be an evolution operator. If there are $r > 0$ and a continuous function $g : [r, \infty) \to (0, \infty)$ such that

$$\inf_{t > r} g(t) < 1,$$

$$\|\Phi(t, t_0)x\| \leq g(t-t_0)\|x\|,$$ for all $t_0 \geq 0$, $t \geq t_0 + r$ and $x \in X_{t_0}$, then $\Phi(\cdot, \cdot)$ is uniformly exponentially stable.
Proof. Let \( \delta > r \) such that \( g(\delta) < 1 \).
For \( t \geq t_0 \geq 0 \) there is \( n \in \mathbb{N} \) such that \( n\delta \leq t - t_0 < (n + 1)\delta \).
Let \( x \in X_{t_0} \). Then

\[
\| \Phi(t, t_0)x \| \leq M e^{\omega(t-n\delta-t_0)} \| \Phi(t_0 + n\delta, t_0)x \| \\
\leq M e^{\omega(t-n\delta-t_0)} g(\delta)^n \| x \|.
\]

Denoting \( \nu = \frac{-\ln g(\delta)}{\delta} > 0 \), it follows that

\[
\| \Phi(t, t_0)x \| \leq M e^{\omega \delta \nu} e^{-\nu(t-t_0)} \| x \|.
\]

Denoting \( N = M e^{(\omega + \nu)\delta} \), we obtain

\[
\| \Phi(t, t_0)x \| \leq N e^{-\nu(t-t_0)} \| x \|,
\]
for \( t \geq t_0 \geq 0 \) and \( x \in X_{t_0} \).

Theorem 1. The evolution operator \( \Phi(\cdot, \cdot) \) is uniformly exponentially stable if and only if there is \( K \in (0, \infty) \) such that

\[
\int_t^\infty \left( \int_u^{u+1} \| \Phi(s, t_0)x \| ds \right) du \leq K \| x \|, \quad \text{for all } t \geq 0 \text{ and } x \in X_t.
\]

Proof. Let \( \Phi(\cdot, \cdot) \) be an evolution operator which satisfy, for a \( K > 0 \), the condition of the hypothesis. We have

\[
\| \Phi(t, t_0)x \| \leq M e^{\omega(t-s)} \| \Phi(s, t_0)x \|,
\]
so

\[
e^{\omega s} \| \Phi(t, t_0)x \| \leq M e^{\omega t} \| \Phi(s, t_0)x \|, \quad \text{for } t \geq s \geq t_0 \geq 0 \text{ and } x \in X_{t_0}.
\]

Let \( t \geq t_0 + 1 \). Integrating successively the last relation we obtain

\[
\frac{1}{\omega}(e^{\omega} - 1)e^{\omega u} \| \Phi(t, t_0)x \| \leq M e^{\omega t} \int_u^{u+1} \| \Phi(s, t_0)x \| ds,
\]
for \( u \in [t_0, t - 1] \), and so

\[
\frac{e^{\omega} - 1}{\omega^2}(e^{\omega t} - e^{\omega t_0}) \| \Phi(t, t_0)x \| \leq
\leq M e^{\omega t} \int_{t_0}^{t-1} \left( \int_u^{u+1} \| \Phi(s, t_0)x \| ds \right) du \leq MK e^{\omega t} \| x \|.
\]
It follows that
\[ e^{-\omega} \| \Phi(t, t_0)x \| \leq e^{-\omega(t-t_0)} \| \Phi(t, t_0)x \| + \frac{MK\omega^2}{\omega^2-1} \| x \| \leq M \left(1 + \frac{K\omega^2}{\omega^2-1}\right) \| x \|. \]

For \( t_0 \leq t < t_0 + 1 \) and \( x \in X_{t_0} \) we have
\[ \| \Phi(t, t_0)x \| \leq M e^{\omega(t-t_0)} \| x \| \leq M e^\omega \| x \|. \]

Denoting \( L = Me^\omega \left(1 + \frac{K\omega^2}{\omega^2-1}\right) \), we obtain
\[ \| \Phi(t, t_0)x \| \leq L \| x \|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}. \]

It follows that, for \( t \geq s \geq t_0 \geq 0 \), \( x \in X_{t_0} \), we have
\[ \| \Phi(t, t_0)x \| = \| \Phi(t, s)\Phi(s, t_0)x \| \leq L \| \Phi(s, t_0)x \|. \]

When \( t \geq t_0 + 1 \), we obtain
\[ \| \Phi(t, t_0)x \| \leq L \int_u^{u+1} \| \Phi(s, t_0)x \| ds, \text{ for all } u \in [t_0, t-1], \]

and so
\[ (t-1-t_0)\| \Phi(t, t_0)x \| \leq L \int_{t_0}^{t-1} \left( \int_u^{u+1} \| \Phi(s, t_0)x \| ds \right) du \leq LK \| x \|. \]

It follows by the preceding lemma that \( \Phi(\cdot, \cdot) \) is uniformly exponentially stable.

The converse is immediately by direct calculation. □

**Theorem 2.** Let \( \Phi(\cdot, \cdot) \) be an evolution operator. If there are \( \alpha > 0 \) and a function \( H(\cdot) : \mathbb{R}_+ \to (0, \infty) \) such that
\[ \int_t^\infty \left( \int_u^{u+1} e^{-\alpha(s-t)} \| \Phi(s, t_0)x \| ds \right) du \leq H(t) \| x \|, \text{ for all } t \geq 0 \text{ and } x \in X_t, \]
then there is a function \( N(\cdot) : \mathbb{R}_+ \to (0, \infty) \) such that
\[ \| \Phi(t, t_0)x \| \leq N(t_0) e^{-\alpha(t-t_0)} \| x \|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}. \]

Hence \( \Phi(\cdot, \cdot) \) will be exponentially stable.

**Proof.** Let \( t_0 \geq 0 \), \( t \geq t_0 + 1 \) and \( x \in X_{t_0} \). We have
\[ \| \Phi(t, t_0)x \| \leq Me^{\omega(t-t_0)} \| \Phi(s, t_0)x \|, \text{ for } s \in [t_0, t]. \]
It follows that
\[ e^{-\alpha t}e^{(\omega + \alpha)s}\|\Phi(t, t_0)x\| \leq Me^{\omega t}e^{\alpha(s-t_0)}\|\Phi(s, t_0)x\|, \]
and by integration, for \( u \in [t_0, t - 1] \), we have
\[ e^{-\alpha t}e^{\omega + \alpha - \frac{1}{\omega + \alpha}}\|\Phi(t, t_0)x\| \leq Me^{\omega t}\int_{t_0}^{t-1} e^{\alpha(s-t_0)}\|\Phi(s, t_0)x\|ds, \]
and so
\[
\left(e^{(\omega + \alpha)(t-1)} - e^{(\omega + \alpha)t_0}\right)\|\Phi(t, t_0)x\| \leq \frac{\left(M(\omega + \alpha)^2}{e^{\omega + \alpha} - 1}H(t_0)\|x\|.
\]
It follows that
\[ e^{\alpha(t-t_0)}\|\Phi(t, t_0)x\| \leq e^{\omega + \alpha}\left(M\frac{(\omega + \alpha)^2}{e^{\omega + \alpha} - 1}H(t_0) + M\right)\|x\|.
\]
Denoting \( N(t_0) = Me^{\omega + \alpha}\left(\frac{(\omega + \alpha)^2}{e^{\omega + \alpha} - 1}H(t_0) + 1\right) \), we obtain
\[ \|\Phi(t, t_0)x\| \leq N(t_0)e^{-\alpha(t-t_0)}\|x\|.
\]
For \( t_0 \leq t < t_0 + 1 \) and \( x \in X_{t_0} \) we have
\[ \|\Phi(t, t_0)x\| \leq Me^{\omega e^{\alpha}(t-t_0)}\|x\|.
\]
So, it follows that
\[ \|\Phi(t, t_0)x\| \leq N(t_0)e^{-\alpha(t-t_0)}\|x\|, \] \text{for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}, \]
Using in the proofs of the theorems the \( p \) power of the norm, respectively the \( p \) power of the inner integral \( (p \in [1, \infty)) \), we obtain the following results.

**Corollary 1.** Let \( \Phi(\cdot, \cdot) \) be an evolution operator and \( p \in [1, \infty) \) be arbitrarily. The following assertions are equivalent.

1) \( \Phi(\cdot, \cdot) \) is uniformly exponentially stable.

2) There is \( K \in (0, \infty) \) such that
\[ \int_{t}^{\infty} \left(\int_{u}^{u+1} \|\Phi(s, t)x\|^p ds \right) du \leq K\|x\|^p, \] \text{for all } t \geq 0 \text{ and } x \in X_t.
3) There is $K \in (0, \infty)$ such that
\[ \int_t^\infty \left( \int_u^{u+1} \| \Phi(s,t)x \|_p ds \right)^p \leq K \| x \|_p, \text{ for all } t \geq 0 \text{ and } x \in X_t. \]

**Corollary 2.** Let $\Phi(\cdot, \cdot)$ be an evolution operator and $p \in [1, \infty)$ be arbitrarily.

1) If there are $\alpha > 0$ and a function $H(\cdot) : \mathbb{R}_+ \to (0, \infty)$ such that
\[ \int_t^\infty \left( \int_u^{u+1} e^{\alpha(s-t)} \| \Phi(s,t)x \|_p ds \right)^p du \leq H(t) \| x \|_p, \text{ for all } t \geq 0 \text{ and } x \in X_t, \]
then there is $N(\cdot) : \mathbb{R}_+ \to (0, \infty)$ such that
\[ \| \Phi(t,t_0)x \| \leq N(t_0)e^{-\frac{\alpha}{p}(t-t_0)} \| x \|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}. \]

2) If there is a function $H(\cdot) : \mathbb{R}_+ \to (0, \infty)$ and $\alpha > 0$ such that
\[ \int_t^\infty \left( \int_u^{u+1} e^{\alpha(s-t)} \| \Phi(s,t)x \| ds \right)^p du \leq H(t) \| x \|_p, \text{ for all } t \geq 0 \text{ and } x \in X_t, \]
then there is $N(\cdot) : \mathbb{R}_+ \to (0, \infty)$ such that
\[ \| \Phi(t,t_0)x \| \leq N(t_0)e^{-\alpha(t-t_0)} \| x \|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}. \]

**References**


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