A NOTE ON STATE ESTIMATION FROM DOUBLY STOCHASTIC POINT PROCESS OBSERVATION

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0. Introduction

In this note we study a state estimation of a Markovian semimartingale from a doubly stochastic point process observation.

All stochastic processes below are supposed to be defined on a filtered probability space \((\Omega, F, (F_t)_{t \geq 0}, P)\) where \((F_t)\) is a filtration satisfying usual conditions.

Consider a state estimation problem where the signal process is a real-valued continuous semimartingale \(X\) that is also a Markov process given by

\[
X_t = X_0 + \int_0^t H_s ds + B_t, \quad t \in \mathbb{R}^+, \tag{0.1}
\]

where \(H_t\) is a continuous process and \(B_t\) is a standard Brownian motion, and the observation is a doubly stochastic point process \(N_t\) driven by \(X_t\): \(N_t\) is a point process of intensity \(\lambda_t = \lambda(X_t)\) where \(\lambda\) is a nonnegative boolean function.

Denote by \(Z^u_t\) the process \(\exp(iuX_t)\). We want to investigate the best state estimation

\[
\pi_t(Z^u_t) = \mathbb{E}[Z^u_t | F_t] \tag{0.2}
\]

where \(F_t\) is the natural filtration of the process \(N_t\) i.e. \(F_t = \sigma(N_s, s \leq t)\). In the sequel the notation \(\pi_t(\ldots)\) stands for the conditional expectation given \(F_t\).

1. A stochastic differential equation for the best state estimation of \(Z^u_t\)

**Theorem 1.** \(\pi_t(Z^u_t)\) satisfies the following equation:

\[
\pi_t(Z^u_t) = E[Z^u_0] + iu \int_0^t \pi_s(Z^u_s H_s) ds - \frac{u^2}{2} \int_0^t \pi_s(Z^u_s) ds + \int_0^t \lambda_s^{-1} \pi_s[(Z^u_s - \pi_s(Z^u_s))(\lambda_s - \pi_s(\lambda_s))](dN_s - \pi_s(\lambda_s)) ds \tag{1.1}
\]
Proof. Applying the Ito formula to $z^u_t = \exp(iuX_t)$ we have

$$Z^u_t = Z^u_0 + \int_0^t \left( iuH_s - \frac{u^2}{2} \right) ds + iu \int_0^t Z^u_s dB_s.$$ 

$Z^u_t$ is in fact a semimartingale, and the filtering equation from point process observation [2] applied to $Z^u_t$:

$$Z_t(Z^u) = E[Z^u_0] + \int_0^t \pi_s \left[ Z^u_s \left( iuH_s - \frac{u^2}{2} \right) \right] ds + \int_0^t \pi_s^{-1}(\lambda)[\pi_s(Z^u_s\lambda_s) - \pi_s(Z^u_s)\pi_s(\lambda_s)][dN_s - \pi_s(\lambda_s)ds].$$

Now

$$\pi_s[[Z^u_s - \pi_s(Z^u_s)][\lambda_s - \pi_s(\lambda_s)]] =$$

$$= \pi_s[Z^u_s\lambda_s - Z^u_s\pi_s(\lambda_s) - \pi_s(Z^u_s)\lambda_s + \pi_s(Z^u_s)\pi_s(\lambda_s)] =$$

$$= \pi_s(Z^u_s\lambda_s) - \pi_s[Z^u_s\pi_s(\lambda_s)] - \pi_s[\pi_s(Z^u_s)\lambda_s] + \pi_s(Z^u_s)\pi_s(\lambda_s). \quad (1.2)$$

It follows from

$$\pi_s[Z^u_s\pi_s(\lambda_s)] = E[Z^u_sE(\lambda_s|F^N_s)]|F^N_s] =$$

$$= E(\lambda_s|F^N_s)E(Z^u_s|F^N_s) = \pi_s(\lambda_s)\pi_s(Z^u_s),$$

and also from

$$\pi_s[\pi_s(Z^u_s)\lambda_s] = \pi_s(Z^u_s)\pi_s(\lambda_s)$$

that it remains only the first and the second terms in the left hand side of (1.2) and we have:

$$\pi_s(Z^u_s\lambda_s) - \pi_s(Z^u_s)\pi_s(\lambda_s) = \pi_s[[Z^u_s - \pi_s(Z^u_s))(\lambda_s - \pi_s(\lambda_s))]$$

and the equation (1.1) is thus completely proved.

Remark. In the multidimensional case, the signal process is a vector process given by

$$X_t = X_0 + \int_0^t H_s ds + B_t,$$

where $X, H, B$ are multidimensional process. By $Z^u_t$ we denote now the process $\exp(i\langle u, X_t \rangle)$, where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $X_t = (X^1_t, \ldots, X^n_t)$ and $\langle \cdot, \cdot \rangle$ stands for the
scalar product in \( \mathbb{R}^n \). And the best state estimation for \( Z_t^u \) based on an observation process that is a doubly stochastic point of intensity \( \lambda_t = \lambda(X_t) \) is

\[
\pi_t(Z_t^u) \equiv E[Z_t^u | \mathcal{F}_t^N] = E[\exp i(u, X_t) | \mathcal{F}_t^N].
\] (1.3)

The stochastic differential equation for \( \pi_t(Z_t^u) \) is the same as (1.1) with \( Z_t^u = \exp(u, X_t) \).

In the next Section, we will establish a connection between the characteristic function of \( X_t \) and the filter of \( Z_t^u \) and so we will see that the laws of the signal \( X_t \) can be completely determined by \( \pi_t(Z_t^u) \).

2. Characteristic function of \( X_t \)

Put

\[
\psi_t(u) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\exp(i u \Delta X_t) - 1 | X_t]
\] (2.1)
is the limit in the right hand side exists, where \( E[|X_t] \) is the conditional expectation given \( X_t \).

Denote by \( \varphi_t(u) \) the characteristic function of \( X_t \):

\[
\varphi_t(u) = E[\exp(iuX_t)] = E[Z_t^u].
\]

We note that

\[
\varphi_{t+\Delta t}(u) = E[\exp(iuX_{t+\Delta t})] = E[\exp iu(X_t + \Delta X_t)] =
\]

\[
= E[\exp(iuX_t \exp iu\Delta X_t)] =
\]

\[
= E[\exp(iuX_t \exp (\exp iu\Delta X_t)X_t)]
\]

\[
\varphi_{t+\Delta t}(u) - \varphi_t(u) = E\{\exp(iuX_t)E[\exp iu\Delta X_t - 1 | X_t]\}
\]

It follows that

\[
\frac{\partial \varphi_t(u)}{\partial t} = \lim_{\Delta t \downarrow 0} E \left\{ (\exp iuX_t) \frac{1}{\Delta t} E[\exp iu\Delta X_t - 1 | X_t] \right\}.
\]

We have now:

\[
\frac{\partial \varphi_t(u)}{\partial t} = E[Z_t^u \psi_t(u)]
\] (2.2)

\[
\varphi_0(u) = E[Z_0^u]
\]
Next, we denote by $\mathcal{F}_{t-\varepsilon}^N$ the $\sigma$-algebra generated by all $N_s$, $s \leq t - \varepsilon$ for all small $\varepsilon > 0$. In noticing that by definition (2.1) $\psi_t(u)$ is conditioning to the random variable $\chi_t$ so it is independent of $\mathcal{F}_{t-\varepsilon}^N$ and we have:

$$E[Z_t^u \psi_t(u)] = E[E(Z_t^u \psi_t(u)|\mathcal{F}_{t-\varepsilon}^N)] = E[\psi_t(u)E(Z_t^u|\mathcal{F}_{t-\varepsilon}^N)].$$

Because of the left continuity of $(\mathcal{F}_{t}^N)$ we have by letting $\varepsilon \to 0$

$$E[Z_t^u \psi_t(u)] = E[\psi_t(u)E(Z_t^u|\mathcal{F}_{t}^N)] = E[\psi_t(u)\pi_t(Z_t^u)]$$

then we have the following

**Proposition 1.** The law of the signal $X_t$ can be determined in term of filtering by the following equation:

$$\frac{\partial \varphi_t(u)}{\partial t} = E[\psi_t(u)\pi_t(Z_t^u)]$$

(2.3)

$$\varphi_0(u) = E[Z_0^u]$$

We will see in next Section that $X_t$ can be recognized by filtering and the process $H_t$.

3. **An expression of the function $\psi_t(u)$**

The equation (1.1) can be rewritten as:

$$dX_t = H_t dt + dB_t$$

(3.1)

or

$$\Delta X_t = H_t \Delta t + \Delta B_t$$

(3.2)

where $\Delta X_t = X_{t+\Delta t} - X_t$, $\Delta B_t = B_{t+\Delta t} - B_t$, $B_t$ is a Brownian motion and since $EB_tB_s = \min(t, s)$, we have

$$E[\exp iu \Delta X_t - 1|X_t] = \exp[iuH_t(X_t)\Delta t]E[\exp iu \Delta B_t|X_t] - 1$$

It follows from the fact that $\Delta B_t$ is normally distributed with mean 0 and covariance $\Delta t$

$$E[iu \Delta B_t|X_t] = \exp \left[ -\frac{1}{2} u^2 \Delta t \right].$$
Hence,

$$\psi_t(u) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\exp iu \Delta X_t - 1|X_t] = \exp \left( iuH_t - \frac{u^2}{2} \right)$$

or

$$\psi_t(u) = \exp \left( iuH_t - \frac{u^2}{2} \right). \quad (3.3)$$

A substitution of this expression of $\psi_t$ into (2.3) yields

Proposition 2.

$$\frac{\partial \phi_t(u)}{\partial t} = E \left\{ \exp \left( iuH_t - \frac{u^2}{2} \right) \pi_t(Z_u^t) \right\}$$

$$\phi_0(u) = E[Z_0] \quad (3.4)$$

4. A Bayes formula for the best state estimation of $Z_u^t$

We know that by a change of reference probability $P \to Q$ such that $P_t \ll Q_t$ for all restriction $P_t$ and $Q_t$ of $P$ and $Q$ respectively to $(\Omega, \mathcal{F}_t)$, we have [1]

$$E_P[U_t|G_t] = \frac{E_Q[U_tL_t|G_t]}{E_Q[L_t|G_t]}$$

where $U_t$ is a real-valued bounded process adapted to $\mathcal{F}_t, G_t$ is any sub $\sigma$-field of $\mathcal{F}_t : G_t \subset \mathcal{F}_t$ and $L_t = \frac{dP_t}{dQ_t}$.

Now, for a doubly stochastic point process $Y_t$ of intensity $\lambda_t = \lambda(X_t)$ we have

$$L_t = \left( \prod_{0 \leq s \leq t} \lambda(X_s) \Delta N_s \right) \exp \left\{ \int_0^t (1 - \lambda(X_s))ds \right\}.$$

We note that under $Q$ the process $N_t$ is a Poisson process of intensity 1. And we have

$$\pi_t(Z_u^t) = \frac{E_Q[Z_u^tL_t|\mathcal{F}_t^N]}{E_Q[L_t|\mathcal{F}_t^N]} = \frac{E_Q[L_t \exp iuX_t|\mathcal{F}_t^N]}{E_Q[L_t|\mathcal{F}_t^N]}.$$

References


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