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A NOTE ON STATE ESTIMATION FROM DOUBLY STOCHASTIC POINT PROCESS OBSERVATION

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0. Introduction

In this note we study a state estimation of a Markovian semimartingale from a doubly stochastic point process observation.

All stochastic processes below are supposed to be defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ where $(\mathcal{F}_t)$ is a filtration satisfying usual conditions.

Consider a state estimation problem where the signal process is a real-valued continuous semimartingale X that is also a Markov process given by

$$X_t = X_0 + \int_0^t H_s ds + B_t, \quad t \in \mathbb{R}^+, \tag{0.1}$$

where H_t is a continuous process and B_t is a standard Brownian motion, and the observation is a doubly stochastic point process N_t driven by X_t: N_t is a point process of intensity $\lambda_t = \lambda(X_t)$ where λ is a nonnegative boolean function.

Denote by Z_t^u the process $\exp(iuX_t)$. We want to investigate the best state estimation

$$\pi_t(Z_t^u) = E[Z_t^u | \mathcal{F}_t^N] \tag{0.2}$$

where $\mathcal{F}_t^N$ is the natural filtration of the process N_t i.e. $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$. In the sequel the notation $\pi_t(\dots)$ stands for the conditional expectation given $\mathcal{F}_t^N$.

1. A stochastic differential equation for the best state estimation of Z_t^u

Theorem 1. *$\pi_t(Z_t^u)$ satisfies the following equation:*

$$\pi_t(Z_t^u) = E[Z_0^u] + iu\int_0^t \pi_s(Z_s^u H_s)ds - \frac{u^2}{2}\int_0^t \pi_s(Z_s^u) +$$

$$+ \int_0^t \lambda_s^{-1}\pi_s[(Z_s^u - \pi_s(Z^u))(\lambda_s - \pi_s(\lambda_s))](dN_s - \pi_s(\lambda_s)ds) \tag{1.1}$$

Proof. Applying the Ito formula to $z_t^u = \exp(iuX_t)$ we have

$$Z_t^u = Z_0^u + \int_0^t \left(iuH_s - \frac{u^2}{2}\right) ds + iu \int_0^t Z_s^u dB_s.$$

Z_t^u is in fact a semimartingale, and the filtering equation from point process observation [2] applied to Z_t^u:

$$Z_t(Z^u) = E[Z_0^u] + \int_0^t \pi_s \left[Z_s^u \left(iuH_s - \frac{u^2}{2}\right)\right] ds+$$

$$+ \int_0^t \pi_s^{-1}(\lambda)[\pi_s(Z^u\lambda_s) - \pi_s(Z_s^u)\pi_s(\lambda_s)][dN_s - \pi_s(\lambda_s)ds].$$

Now

$$\pi_s\{[Z_s^u - \pi_s(Z_s^u)][\lambda_s - \pi_s(\lambda_s)]\} =$$

$$= \pi_s[Z_s^u\lambda_s - Z_s^u\pi_s(\lambda_s) - \pi_s(Z_s^u)\lambda_s + \pi_s(Z_s^u)\pi_s(\lambda_s)] =$$

$$= \pi_s(Z_s^u\lambda_s) - \pi_s[Z_s^u\pi_s(\lambda_s)] - \pi_s[\pi_s(Z_s^u)\lambda_s] + \pi_s(Z_s^u)\pi_s(\lambda_s). \tag{1.2}$$

It follows from

$$\pi_s[Z_s^u\pi_s(\lambda_s)] = E[Z_s^u E(\lambda_s|\mathcal{F}_s^N)|\mathcal{F}_s^N] =$$

$$= E(\lambda_s|\mathcal{F}_s^N)E(Z_s^u|\mathcal{F}_s^N) = \pi_s(\lambda_s)\pi_s(Z^u),$$

and also from

$$\pi_s[\pi_s(Z_s^u)\lambda_s] = \pi_s(Z_s^u)\pi_s(\lambda_s)$$

that it remains only the first and the second terms in the left hand side of (1.2) and we have:

$$\pi_s(Z_s^u\lambda_s) - \pi_s(Z_s^u)\pi_s(\lambda_s) = \pi_s[(Z_s^u - \pi_s(Z_s^u))(\lambda_s - \pi_s(\lambda_s))]$$

and the equation (1.1) is thus completely proved.

Remark. In the multidimensional case, the signal process is a vector process given by

$$X_t = X_0 + \int_0^t H_s ds + B_t$$

where X, H, B are multidimensional process. By Z_t^u we denote now the process $\exp(i\langle u, X_t\rangle)$, where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $X_t = (X_t^1, \ldots, X_t^n)$ and $\langle , \rangle$ stands for the scalar product in $\mathbb{R}^n$. And the best state estimation for Z_t^u based on an observation process that is a doubly stochastic point of intensity $\lambda_t = \lambda(X_t)$ is

$$\pi_t(Z_t^u) \equiv E[Z_t^u|\mathcal{F}_t^N] = E[\exp i\langle u, X_t\rangle|\mathcal{F}_t^N]. \tag{1.3}$$

The stochastic differential equation for $\pi_t(Z_t^u)$ is the same as (1.1) with $Z_t^u = \exp\langle u, X_t\rangle$.

In the next Section, we will establish a connection between the characteristic function of X_t and the filter of Z_t^u and so we will see that the laws of the signal X_t can be completely determined by $\pi_t(Z_t^u)$.

2. **Characteristic function of X_t**

Put

$$\psi_t(u) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} E[\exp(iu\Delta X_t) - 1|X_t] \tag{2.1}$$

is the limit in the right hand side exists, where $E[\cdot|X_t]$ is the conditional expectation given X_t.

Denote by $\varphi_t(u)$ the characteristic function of X_t:

$$\varphi_t(u) = E[\exp(iuX_t)] = E[Z_t^u].$$

We note that

$$\varphi_{t+\Delta t}(u) = E[\exp(iuX_{t+\Delta t})] = E[\exp iu(X_t + \Delta X_t)] =$$

$$= E[\exp(iuX_t \exp iu\Delta X_t] =$$

$$= E[\exp(iuX_t E(\exp iu\Delta X_t|X_t))]$$

$$\varphi_{t+\Delta t}(u) - \varphi_t(u) = E\{(\exp(iuX_t)E[\exp iu\Delta X_t - 1|X_t]\}$$

It follows that

$$\frac{\partial \varphi_t(u)}{\partial t} = \lim_{\Delta t\downarrow 0} E\left\{(\exp iuX_t)\frac{1}{\Delta t}E[\exp iu\Delta X_t - 1|X_t]\right\}.$$

We have now:

$$\frac{\partial \varphi_t(u)}{\partial t} = E[Z_t^u \psi_t(u)]$$
$$\varphi_0(u) = E[Z_0^u] \tag{2.2}$$

Next, we denote by $\mathcal{F}^N_{t-\varepsilon}$ the σ-algebra generated by all N_s, $s \leq t-\varepsilon$ for all small $\varepsilon > 0$. In noticing that by definition (2.1) $\psi_t(u)$ is conditioning to the random variable χ_t so it is independent of $\mathcal{F}^N_{t-\varepsilon}$ and we have:

$$E[Z^u_t \psi(u)] = E[E(Z^u_t \psi(u)|\mathcal{F}^N_{t-\varepsilon})] = E[\psi_t(u)E(Z^u_t|\mathcal{F}^N_{t-\varepsilon})].$$

Because of the left continuity of $(\mathcal{F}^N_t)$ we have by letting $\varepsilon \to 0$

$$E[Z^u_t \psi_t(u)] = E[\psi_t(u)E(Z^u_t|\mathcal{F}^N_t)] = E[\psi_t(u)\pi_t(Z^u_t)]$$

then we have the following

Proposition 1. *The law of the signal X_t can be determined in term of filtering by the following equation:*

$$\begin{aligned} \frac{\partial \varphi_t(u)}{\partial t} &= E[\psi_t(u)\pi_t(Z^u_t)] \\ \varphi_0(u) &= E[Z^u_0] \end{aligned} \tag{2.3}$$

We will see in next Section that X_t can be recognized by filtering and the process H_t.

3. An expression of the function $\psi_t(u)$

The equation (1.1) can be rewritten as:

$$dX_t = H_t dt + dB_t \tag{3.1}$$

or

$$\Delta X_t = H_t \Delta t + \Delta B_t \tag{3.2}$$

where $\Delta X_t = X_{t+\Delta t} - X_t$, $\Delta B_t = B_{t+\Delta t} - B_t$, B_t is a Brownian motion and since $EB_tB_s = \min(t,s)$, we have

$$E[\exp iu\Delta X_t - 1|X_t] = \exp[iuH_t(X_t)\Delta t]E[\exp iu\Delta B_t|X_t] - 1$$

It follows from the fact that ΔB_t is normally distributed with mean 0 and covariance Δt

$$E[iu\Delta B_t|X_t] = \exp\left[-\frac{1}{2}u^2\Delta t\right].$$

Hence,

$$\psi_t(u) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\exp iu\Delta X_t - 1|X_t] =$$
$$= \exp\left(iuH_t - \frac{u^2}{2}\right)$$

or

$$\psi_t(u) = \exp\left(iuH_t - \frac{u^2}{2}\right). \tag{3.3}$$

A substitution of this expression of ψ_t into (2.3) yields

Proposition 2.

$$\frac{\partial \varphi_t(u)}{\partial t} = E\left\{\left[\exp\left(iuH_t - \frac{u^2}{2}\right)\right]\pi_t(Z_t^u)\right\} \tag{3.4}$$
$$\varphi_0(u) = E[Z_0]$$

4. A Bayes formula for the best state estimation of Z_t^u

We know that by a change of reference probability $P \to Q$ such that $P_t \ll Q_t$ for all restriction P_t and Q_t of P and Q respectively to $(\Omega, \mathcal{F}_t)$, we have [1]

$$E_P[U_t|\mathcal{G}_t] = \frac{E_Q[U_t L_t|\mathcal{G}_t]}{E_Q[L_t|\mathcal{G}_t]}$$

where U_t is a real-valued bounded process adapted to $\mathcal{F}_t, \mathcal{G}_t$ is any sub σ-field of $\mathcal{F}_t : \mathcal{G}_t \subset \mathcal{F}_t$ and $L_t = \dfrac{dP_t}{dQ_t}$.

Now, for a doubly stochastic point process Y_t of intensity $\lambda_t = \lambda(X_t)$ we have

$$L_t = \left(\prod_{0 \le s \le t} \lambda(X_s)\Delta N_s\right) \exp\left\{\int_0^t (1 - \lambda(X_s))ds\right\}.$$

We note that under Q the process N_t is a Poisson process of intensity 1. And we have

$$\pi_t(Z_t^u) = \frac{E_Q[Z_t^u L_t|\mathcal{F}_t^N]}{E_Q[L_t|\mathcal{F}_t^N]} = \frac{E_Q[L_t \exp iuX_t|\mathcal{F}_t^N]}{E_Q[L_t|\mathcal{F}_t^N]}.$$

References

[1] Alan F. Karr, *Point Processes and Their Statistical Inference*, Second Edition, Marcel Dekker, Inc, 1991.

[2] Tran Hung Thao, *Note on Filtering from Point Processes*, Acta Mathematica, Volume 16, No.1, 1991, 39-47.

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