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COVERS OF GRAPHS BY TWO CONVEX SETS

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Abstract. The nontrivial convex 2-cover problem of a simple graph is studied. We establish the existence of a convex $(2, nt)$-cover in dependency of existing convex $(2, t)$-covers. We prove that it is NP-complete to decide whether a graph that has convex $(2, t)$-covers also has a convex $(2, nt)$-cover. In addition, we identify some classes of graphs for which there exists a convex $(2, nt)$-cover.

1. Introduction

In this work we consider only simple connected graphs. We denote by $G = (X; U)$ a graph with vertex set X, $|X| = n$, and edge set U, $|U| = m$. The *neighborhood* of a vertex $x \in X$ is the set of all vertices $y \in X$ such that $x \sim y$, and it is denoted by $\Gamma(x)$.

The distance between vertices x and y in G is denoted by $d(x, y)$. We say that $x \in X$ is a *simplicial* vertex of G if $\Gamma(x)$ is a clique.

Let us remind some notions defined in [1]: a) *metric segment* $\langle x, y\rangle$ is the set of all vertices lying on a shortest path between vertices $x, y \in X$; b) a set $S \subseteq X$ is called *convex* if $\langle x, y\rangle \subseteq S$ for all $x, y \in S$; c) *convex hull* of $S \subseteq X$, denoted $d - conv(S)$, is the smallest convex set containing S.

A set $S \subseteq X$ is called *nontrivial* if $3 \leq |S| \leq n - 1$. Otherwise S is called *trivial*.

A family of sets $\mathcal{P}_2(G) = \{X_1, X_2\}$ is called *convex* 2*-cover* of the graph $G = (X; U)$ if $X_1 \nsubseteq X_2$, $X_2 \nsubseteq X_1$ and $X_1 \cup X_2 = X$, where X_1 and X_2 are convex sets in G. The concept of *convex p-cover* of a graph for $p \geq 2$ is defined in [2] as a cover of graph by p convex sets. In particular, $\mathcal{P}_2(G)$ is called *convex* 2*-partition* of graph G if it is a convex 2-cover of G and sets of $\mathcal{P}_2(G)$ are disjoint.

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$\mathcal{P}_{2,t}(G) = \{S_t, S_{nt}\}$ is said to be a *convex* $(2,t)$*-cover* of G if it is a convex 2-cover of G such that S_t is a trivial set. In the same way, $\mathcal{P}_{2,nt}(G) = \{S_1, S_2\}$ is said to be a *convex* $(2,nt)$*-cover* of G if it is a convex 2-cover of G such that S_1 and S_2 are nontrivial.

Denote by $\widetilde{\mathcal{P}}_{2,t}(G) = \{\mathcal{P}^1_{2,t}, \mathcal{P}^2_{2,t}, \dots, \mathcal{P}^k_{2,t}\}$, $k \in N$, a family of all possible convex $(2,t)$-covers of G.

Deciding if a graph has a convex 2-cover was declared an open problem in [2]. After, we proved its NP-completeness [7]. We know that verifying if a set is convex can be done in polynomial time [4]. Consequently, determining if there exists a convex $(2,t)$-cover also can be done in polynomial time. Thus, it is NP-complete to decide whether a graph G has a convex $(2,nt)$-cover.

This paper is organized as follows. In section 2 we establish the existence of a convex $(2,nt)$-cover in dependency on existing convex $(2,t)$-covers. Also, identification algorithms for some specifical graph classes are developed. In section 3 we prove that it is NP-complete to decide whether a graph that has convex $(2,t)$-covers also has a convex $(2,nt)$-cover. In section 4 we present some graph classes, which have a convex $(2,nt)$-cover.

2. Convex $(2,nt)$-cover via convex $(2,t)$-covers

It is clear that every simple connected graph G on n vertices, where $n = 2$ or $n = 3$, has a convex $(2,t)$-cover but has no a convex $(2,nt)$-cover.

Let us analyze the case $n = 4$.

Consider a cycle on 4 vertices C_4 and the *nontrivial convex cover number* $\varphi_{cn}(G)$ as the least integer $p \geq 2$ for which G has a convex p-cover by nontrivial convex sets. The next theorem is true.

Theorem 2.1. [7] *If G is a simple connected graph on 4 vertices, then $\varphi_{cn}(G) = 2$ if and only if $G \neq C_4$.*

As a consequence of Theorem 2.1, we get the following result.

Corollary 2.2. *Let G be a simple connected graph on 4 vertices. Then G has a convex $(2,nt)$-cover if and only if $G \neq C_4$.*

According to definition of the nontrivial convex cover number, Corollary 2.2 is true.

In the sequel we analyze the case $n \geq 5$.

Theorem 2.3. *Let $G = (X; U)$, $|X| \geq 5$, be a simple connected graph. Then the following conditions are equivalent:*

1) *in G there exists a simplicial vertex $x \in X$;*
2) *in G there exists $\mathcal{P}_{2,t}(G) = \{S_t = \{x\}, S_{nt} = X \backslash \{x\}\}$;*
3) *in G there exists $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x\}\}$.*

Proof. Since x is a simplicial vertex in G, it follows that every two vertices $y, z \in \Gamma(x)$ are adjacent. Further, $d - conv(\Gamma(x)) = \Gamma(x)$ and $d - conv(X \backslash \{x\}) = X \backslash \{x\}$. Thus, G can be covered by a convex $(2,t)$-cover:

$$\mathcal{P}_{2,t}(G) = \{S_t = \{x\}, S_{nt} = X \backslash \{x\}\}.$$

Consequently 1) $\Rightarrow$ 2).

Suppose there exists a convex $(2,t)$-cover $\mathcal{P}_{2,t}(G) = \{S_t = \{x\}, S_{nt} = X \backslash \{x\}\}$. Graph G is connected. Hence, there is at least one vertex y such that $y \sim x$ and $d - conv(\{x,y\}) = \{x,y\}$. Therefore, G can be covered by a convex $(2,t)$-cover:

$$\mathcal{P}_{2,t}(G) = \{S_t = \{x,y\}, S_{nt} = X \backslash \{x\}\}.$$

Consequently 2) $\Rightarrow$ 3).

Suppose there exists a convex $(2,t)$-cover $\mathcal{P}_{2,t}(G) = \{S_t = \{x,y\}, S_{nt} = X \backslash \{x\}\}$. Since S_{nt} is convex, $\Gamma(x)$ is a clique in G. Whence x is a simplicial vertex. Consequently 3) $\Rightarrow$ 1). □

Theorem 2.4. *Let $G = (X; U)$, $|X| \geq 5$, be a simple connected graph that contains a simplicial vertex. Then G has a convex $(2, nt)$-cover.*

Proof. It follows from Theorem 2.3 that there is a convex $(2,t)$-cover $\mathcal{P}_{2,t}(G) = \{S_t = \{x\}, S_{nt} = X \backslash \{x\}\}$ such that x is a simplicial vertex. We consider 2 cases.

1) $\Gamma(x) = \{y\}$. Since G is a connected graph and $|X| - 1 > 3$, there exists $z \in S_{nt}$ such that $z \sim y$. Taking into account that $\langle x, z \rangle = \{x,y,z\}$ and $d - conv(\{x,y,z\}) = \{x,y,z\}$, we obtain the nontrivial convex set $\{x,y,z\}$. This yields that G has a convex $(2,nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x,y,z\}, S_2 = S_{nt}\}.$$

2) $|\Gamma(x)| \geq 2$. Select two vertices $y, z \in \Gamma(x)$. Since x is a simplicial vertex, $y \sim z$ and $\{x,y,z\}$ is a triangle that is a nontrivial convex set. This implies that G has a convex $(2,nt)$-cover $\mathcal{P}_{2,nt}(G) = \{S_1 = \{x,y,z\}, S_2 = S_{nt}\}$.

Finally, G has a convex $(2,nt)$-cover. □

Theorem 2.5. *Let $G = (X; U)$, $|X| \geq 5$, be a simple connected graph without simplicial vertices. Then the following conditions are equivalent:*

1) *in G there exist two adjacent vertices $x, y \in X$ such that $A = \Gamma(x) \backslash \{y\}$ and $B = \Gamma(y) \backslash \{x\}$ are cliques in G, where for all vertices $a \in A$, $b \in B$, the inequality $d(a,b) \leq 2$ is satisfied;*
2) *in G there exists a convex $(2,t)$-cover $\mathcal{P}_{2,t}(G) = \{S_t = \{x,y\}, S_{nt} = X \backslash \{x,y\}\}$.*

Proof. Combining Theorem 2.3 with the absence of simplicial vertices in G, we get that G has no a convex $(2,t)$-cover such that cardinality of the trivial convex set is one, or cardinality of the trivial convex set is two and trivial set intersects nontrivial set.

Let $x, y \in X$ be two vertices, which satisfy the condition 1). Then the following relations are true:

$$d - conv(\{x, y\}) = \{x, y\},\ \{x, y\} \cap d - conv(A \cap B) = \emptyset.$$

It follows that G has a convex $(2, t)$-cover:

$$\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x, y\}\}.$$

Consequently 1) $\Rightarrow$ 2)

Suppose there exists a convex $(2, t)$-cover $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x, y\}\}$. According to the theorem conditions G does not contain simplicial vertices. Because of the connectivity of S_t and S_{nt}, we have $x \sim y$, and sets $A = \Gamma(x) \backslash \{y\}$, $B = \Gamma(y) \backslash \{x\}$ generate cliques in G. Moreover, if there exist two vertices $a \in A$, $b \in B$ such that $d(a, b) > 2$, then $\{x, y\} \subseteq \langle a, b\rangle \subseteq S_{nt}$. This contradicts convexity of S_{nt}. Further, this means that for all vertices $a \in A$, $b \in B$, we have $d(a, b) \leq 2$. Consequently 2) $\Rightarrow$ 1). □

Theorem 2.6. *Let $G = (X; U)$, $|X| \geq 5$, be a simple connected graph without simplicial vertices and let $\widetilde{\mathcal{P}}_{2,t}(G)$ contains two convex $(2, t)$-covers such that intersection of their trivial convex sets is empty. Then G has a convex $(2, nt)$-cover.*

Theorem 2.6 follows directly from the fact that the nontrivial convex sets of respective convex $(2, t)$-covers form a convex $(2, nt)$-cover of G.

Theorem 2.7. *Let $G = (X; U)$, $|X| \geq 5$, be a simple connected graph without simplicial vertices and let $|\widetilde{\mathcal{P}}_{2,t}(G)| = k \geq 2$ such that intersection of trivial sets S_t^i, $1 \leq i \leq k$, of any two convex $(2, t)$-covers is not empty. Then exactly one of the following conditions is satisfied:*

1) $|\widetilde{\mathcal{P}}_{2,t}(G)| = 3$ *and* $S_t^1 \cup S_t^2 \cup S_t^3$ *generates a triangle in* G*;*
2) $|\bigcap_{i=1}^{k} S_t^i| = 1$.

Proof. G has no simplicial vertices. Further, using Theorem 2.3, we get that cardinality of trivial convex set for all convex $(2, t)$-covers of G is two and trivial convex set does not intersect nontrivial convex set. Let us consider 3 cases.

$|\widetilde{\mathcal{P}}_{2,t}(G)| = 2$. It follows that $|S_t^1 \cap S_t^2| = 1$. Hence, condition 2) is satisfied.

$|\widetilde{\mathcal{P}}_{2,t}(G)| = 3$. If $|S_t^1 \cap S_t^2 \cap S_t^3| = 1$, then condition 2) is satisfied. Otherwise $S_t^1 \cup S_t^2 \cup S_t^3$ generates a triangle in G and condition 1) is satisfied.

$|\widetilde{\mathcal{P}}_{2,t}(G)| \geq 4$. Obviously, in this case we have $|\bigcap_{i=1}^{|\widetilde{\mathcal{P}}_{2,t}(G)|} S_t^i| = 1$. This means that condition 2) is satisfied. □

Theorem 2.8. *Let $G = (X; U)$, $|X| \geq 5$, be a simple connected graph, without simplicial vertices, that satisfies the equality:*

$$\widetilde{\mathcal{P}}_{2,t}(G) = \{\mathcal{P}^i_{2,t}(G) = \{S^i_t, S^i_{nt}\} : 1 \leq i \leq 3\},$$

where $S^1_t \cup S^2_t \cup S^3_t$ generates a triangle in G. Then G has a convex $(2, nt)$-cover.

Proof. Denote $S = S^1_t \cup S^2_t \cup S^3_t$. It is obvious that G can be covered by one of the three convex $(2, nt)$-covers:

$$\mathcal{P}^1_{2,nt}(G) = \{S^1_{nt}, S\},\ \mathcal{P}^2_{2,nt}(G) = \{S^2_{nt}, S\},\ \mathcal{P}^3_{2,nt}(G) = \{S^3_{nt}, S\}.$$

This proves the theorem. □

Theorem 2.9. *Let $G = (X; U)$, $|X| \geq 5$, be a simple connected graph, without simplicial vertices, that satisfies the equality:*

$$\widetilde{\mathcal{P}}_{2,t}(G) = \{\mathcal{P}^i_{2,t}(G) = \{S^i_t = \{a, b_i\}, S^i_{nt}\} : 1 \leq i \leq k,\ k \geq 3\}.$$

Then G has a convex $(2, nt)$-cover.

Proof. According to the theorem conditions, we have $|\widetilde{\mathcal{P}}_{2,t}(G)| \geq 3$, $\bigcap_{i=1}^k S^i_t = \{a\}$ and $|\Gamma(a) \backslash \{b_i\}| \geq 2$ for $1 \leq i \leq k$. Sets S^i_{nt}, $1 \leq i \leq k$, are convex nontrivial due to inequality $|X| \geq 5$. Since, combining absence of simplicial vertices in G with Theorem 2.3, we obtain that G has only convex $(2, t)$-covers such that the cardinality of the trivial convex set is two and trivial convex set does not intersect nontrivial convex set. Now, $b_i \sim b_j$ for all $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, because $a \notin S^i_{nt}$, $1 \leq i \leq k$. Therefore, $\bigcup_{i=1}^k S^i_t$ is a nontrivial clique in G. Thus, $\bigcup_{i=1}^k S^i_t$ is a nontrivial convex set. Finally, there is one of possible convex $(2, nt)$-covers of graph G:

$$\mathcal{P}_{2,nt}(G) = \{\{a, b_1, b_2\}, S^1_{nt}\}.$$

This proves the theorem. □

Now we give the definition of the graph family $\mathcal{F}$, which will be useful in the sequel.

Define $\mathcal{F}$ as the family of graphs $G = (X;\ U)$ that satisfy the following conditions:

a) $X = \{a, b_1, b_2, x_1, x_2, \ldots, x_m\}$, $m \geq 1$;
b) $U = \{\{a, b_1\}, \{a, b_2\}\} \cup \{\{x_i, x_j\} : 1 \leq i, j \leq m; i \neq j\} \cup \{\{b_1, x_i\}, \{b_2, x_i\} : 1 \leq i \leq m\}$.

It can easily be checked that all graphs $G \in \mathcal{F}$ on $n \geq 5$ vertices have exactly two convex $(2, t)$-covers:

$$\mathcal{P}^1_{2,t}(G) = \{\{a, b_1\}, \{b_2, x_1, \ldots, x_m\}\},\ \mathcal{P}^2_{2,t}(G) = \{\{a, b_2\}, \{b_1, x_1, \ldots, x_m\}\}.$$

Graph family $\mathcal{F}$ is presented in Figure 1.

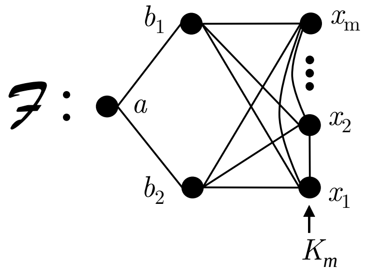

FIGURE 1. Graph family $\mathcal{F}$

Theorem 2.10. *A graph $G = (X;\, U) \in \mathcal{F}$ has no a convex $(2, nt)$-cover.*

Proof. By definition, $|X| \geq 4$. If $|X| = 4$, then $G = C_4$. Under the conditions of Corollary 2.2, G has no a convex $(2, nt)$-cover.

Suppose $|X| \geq 5$. Assume that graph G has a convex $(2, nt)$-cover. Further, one of nontrivial convex sets of this convex $(2, nt)$-cover contains vertices $\{b_1, b_2\}$ or $\{a, x\}$, where $x \in X\backslash\{a, b_1, b_2\}$. Notice that for every graph $G = (X;\, U) \in \mathcal{F}$ the following conditions hold:

$$\{b_1, b_2\} \subseteq \langle a, x\rangle, \text{ for all } x \in X\backslash\{a, b_1, b_2\};$$

$$d - conv(\{b_1, b_2\}) = X.$$

This contradiction proves the theorem. □

Theorem 2.11. *Let $G = (X;\, U)$, $|X| \geq 5$, $G \notin \mathcal{F}$, be a simple connected graph, without simplicial vertices, that satisfies the equality:*

$$\widetilde{\mathcal{P}}_{2,t}(G) = \{\mathcal{P}^1_{2,t}(G) = \{S^1_t = \{a, b_1\}, S^1_{nt}\}, \mathcal{P}^2_{2,t}(G) = \{S^2_t = \{a, b_2\}, S^2_{nt}\}\}.$$

Then G has a convex $(2, nt)$-cover.

Proof. Suppose $b_1 \sim b_2$. Then G has convex $(2, nt)$-covers:

$$\mathcal{P}^1_{2,nt}(G) = \{\{a, b_1, b_2\}, S^1_{nt}\}, \mathcal{P}^2_{2,nt}(G) = \{\{a, b_1, b_2\}, S^2_{nt}\}.$$

Now suppose that $b_1 \nsim b_2$. Denote $A = \Gamma(a)\backslash\{b_1\}$, $B = \Gamma(b_1)\backslash\{a\}$. We see that $A, B \subseteq S^1_{nt}$. If $S^1_{nt} \neq d - conv(A \cup B)$, then G has a convex $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{a, b_1\} \cup d - conv(A \cup B), S_2 = S^1_{nt}\}.$$

Assume that $S^1_{nt} = d - conv(A \cup B)$. In addition, suppose that $|A| \geq 2$. It follows from Theorem 2.5 that $A \cup \{a\}$ is a clique in G. Thus, $A \cup \{a\}$ is a nontrivial convex set. By the theorem conditions, we have $b_2 \in A$, $b_1 \in S^2_{nt}$. Hence, G has a convex $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = A \cup \{a\}, S_2 = S^2_{nt}\}.$$

Further assume that $A = b_2$. In accordance with Theorem 2.5, we obtain that $B \geq 1$. Let us consider 2 cases.

Suppose $S^1_{nt} \neq A \cup B$. Then, combining convexity of S^1_{nt} with Theorem 2.5, there is a vertex $x \in B$ that satisfies $d(b_2, x) = 2$ such that there is a vertex $y \in \langle b_2, x\rangle$, where $y \notin A \cup B$, $y \in d - conv(A \cup B)$, otherwise $S^1_{nt} = A \cup B$. This implies that G has a convex $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{a, b_1, x\}, S_2 = S^1_{nt}\}.$$

Suppose $S^1_{nt} = A \cup B$. Then, since $|X| \geq 5$ and $|A| = 1$, it follows that $|B| \geq 2$. If $b_2 \sim x$ for all $x \in B$, then $G \in \mathcal{F}$ and by Theorem 2.10, it follows that this graph has no a convex $(2, nt)$-cover. Conversely, graph G has a convex $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = d - conv(\{b_1, b_2\}), S_2 = B \cup \{b_1\}\}.$$

The theorem is proved. □

Let us remark that every simple connected graph, that contains simplicial vertices, has at least two different convex $(2, t)$-covers. This follows directly from Theorem 2.3.

Now we define some families of graphs.

By $\mathcal{J}$ denote a family of simple connected graphs on $n \geq 5$ vertices that have at least two different convex $(2, t)$-covers and not belong to $\mathcal{F}$.

By $\mathcal{H}$ denote a family of simple connected graphs on $n \geq 5$ vertices that have exactly one convex $(2, t)$-cover.

Theorem 2.12. *A graph $G \in \mathcal{J}$ has a convex $(2, nt)$-cover.*

Theorem 2.12 follows directly from Theorems 2.3 - 2.11.

Let $\mathcal{H}'$ be a subfamily of $\mathcal{H}$ with the following properties:

a) $A \cap B = \emptyset$, where $A = \Gamma(x)\backslash\{y\}$, $B = \Gamma(y)\backslash\{x\}$ such that $\{x, y\}$ is the trivial set of the convex $(2, t)$-cover of a graph;
b) For each $a \in A$ there exists $b \in B$ such that $a \sim b$ and for each $b \in B$ there exists $a \in A$ such that $b \sim a$;
c) $d - conv(A \cup B) = S_{nt}$, where S_{nt} is the nontrivial set of the convex $(2, t)$-cover of a graph;
d) $S_{nt} \neq A \cup B$. This implies that there exist $a \in A$, $b \in B$, $c \in C$ such that $d(a, b) = 2$ and $c \in \langle a, b\rangle$, where $C = S_{nt}\backslash(A \cup B)$.

Let $\mathcal{H}'' = \mathcal{H}\backslash\mathcal{H}'$.

Theorem 2.13. *A graph $G \in \mathcal{H}''$ has a convex $(2, nt)$-cover.*

Proof. Let $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt}\}$ be a convex $(2, t)$-cover of G. Denote $A = \Gamma(x)\backslash\{y\}$, $B = \Gamma(y)\backslash\{x\}$. Since $G \in \mathcal{H}''$, where $\mathcal{H}'' = \mathcal{H}\backslash\mathcal{H}'$, we have $G \notin \mathcal{H}'$ and it follows that at least one property that characterize the family $\mathcal{H}'$ is not satisfied.

If $A \cap B \neq \emptyset$, then G has a convex $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x, y, z\}, S_2 = S_{nt}\},$$

where $z \in A \cap B$.

Assume that the property a) is satisfied. Conversely, by the above, G has a convex $(2, nt)$-cover. If there exists $a \in A$ for which does not exist $b \in B$ such that $a \sim b$, then G has a convex a $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x, y, a\}, S_2 = S_{nt}\}.$$

In the same way, if there exists $b \in B$ for which does not exist $a \in A$ such that $b \sim a$, then G has a convex a $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x, y, b\}, S_2 = S_{nt}\}.$$

If $d - conv(A \cup B) \neq S_{nt}$, then G has a convex $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x, y\} \cup d - conv(A \cup B), S_2 = S_{nt}\}.$$

If $S_{nt} = A \cup B$. Then we consider two cases.

1) Suppose $|A| \geq 2$ and $|B| \geq 2$. Then G has a convex $(2, nt)$-cover:

$$\mathcal{P}_{2,nt}(G) = \{S_1 = A \cup \{x\}, S_2 = B \cup \{y\}\}.$$

2) Suppose $|A| = 1$. Since every graph of the family $\mathcal{H}''$ has at least five vertices, we get $|B| \geq 2$. Assume that the properties a) and b) are satisfied. Conversely, by the above, G has a convex $(2, nt)$-cover. Let $A = \{v\}$. According to the property b), the vertex v is adjacent to all vertices of B and further $G \in \mathcal{F}$. By definition, $\mathcal{H}''$ is the family of graphs that have exactly one convex $(2, t)$-cover but every graph that belongs to the family $\mathcal{F}$ has exactly two convex $(2, t)$-covers. This implies a contradiction. Similarly, we get a contradiction if suppose $|B| = 1$. Thus, $|A| \geq 2$ and $|B| \geq 2$ but in this case G has a convex $(2, nt)$-cover. □

Consider simple connected graph G has n vertices and m edges. In the sequel, we present some algorithms that determine appartenance of G to the classes: $\mathcal{F}$, $\mathcal{J}$, $\mathcal{H}'$, $\mathcal{H}''$.

Next we propose the Algorithm 2.14 that determine whether a graph G belongs to the family $\mathcal{F}$.

Algorithm 2.14.

Input: *Simple connected graph* $G = (X; U)$.
Output: *YES:* G *belongs to* $\mathcal{F}$, *or NO:* G *does not belong to* $\mathcal{F}$.
Step 1) If $|X| \leq 3$, *then return NO.*
Step 2) If $|X| = 4$, *then check whether* $G = C_4$. *If* $G = C_4$, *then return YES; otherwise return NO.*
Step 3) Check whether there exists or not a unique vertex $x \in X$ *such that* $\Gamma(x) = \{y, z\}$ *and* $y \nsim z$. *If not, then return NO.*
Step 4) Check whether both $\{y\} \cup X \backslash \{x, z\}$ *and* $\{z\} \cup X \backslash \{x, y\}$ *are cliques in* G. *If so, then return YES; otherwise return NO.*

Theorem 2.15. *It can be decided in time* $O(n^2)$ *whether a graph* G *belongs to the family* $\mathcal{F}$.

Proof. Evidently, steps 1) and 2) run in constant time. The step 3) is executed in $O(n)$ time. It is clear that it can be verified in $O(n^2)$ time if the given subgraph is a clique or not. Hence the step 4) operates in $O(n^2)$. Based on the mentioned facts, the execution time of the algorithm is $O(n^2)$. □

Algorithm 2.16 determines whether or not a graph G belongs to one of the families: $\mathcal{J}$, $\mathcal{H}'$, $\mathcal{H}''$.

Algorithm 2.16.

Input: *Simple connected graph* $G = (X; U)$.
Output: $F\mathcal{J}$: G *belongs to* $\mathcal{J}$, *or* $F\mathcal{H}'$: G *belongs to* $\mathcal{H}'$, *or* $F\mathcal{H}''$: G *belongs to* $\mathcal{H}''$, *or NO:* G *does not belong to any of the families.*
Step 1) Apply Algorithm 2.14. If Algorithm 2.14 returns YES, then return NO.
Step 2) Check whether there exists or not a simplicial vertex in G. *If there is a simplicial vertex in* G, *then return* $F\mathcal{J}$.
Step 3) Search all convex $(2, t)$*-covers of* G, *i.e., define* $\widetilde{\mathcal{P}}_{2,t}(G)$. *For this purpose search all adjacent vertices* $x, y \in X$, *which satisfy the next equality* $d - conv(X \backslash \{x, y\}) = X \backslash \{x, y\}$.
Step 4) If $\widetilde{\mathcal{P}}_{2,t}(G) = \emptyset$, *then return NO.*
Step 5) If $|\widetilde{\mathcal{P}}_{2,t}(G)| \geq 2$, *then return* $F\mathcal{J}$.
Step 6) If $A \cap B \neq \emptyset$ *such that* $A = \Gamma(x) \backslash \{y\}$, $B = \Gamma(y) \backslash \{x\}$, *where* $\{x, y\}$ *is the trivial set of the single convex* $(2, t)$*-cover of* $\widetilde{\mathcal{P}}_{2,t}(G)$, *then return* $F\mathcal{H}''$.
Step 7) Check whether there exist $a \in A$ *such that, for all* $b \in B$ *the condition* $a \nsim b$ *is satisfied or there exist* $b \in B$ *such that, for all* $a \in A$ *the condition* $b \nsim a$ *is satisfied. If there exists such* $a \in A$ *or* $b \in B$, *then return* $F\mathcal{H}''$.

Step 8) Compute $d-conv(A \cup B)$. If $d-conv(A \cup B) \neq S_{nt}$, where S_{nt} is the nontrivial set of the single convex $(2,t)$-cover of $\widetilde{\mathcal{P}}_{2,t}(G)$, then return $F\mathcal{H}''$.

Step 9) If $S_{nt} = A \cup B$, then return $F\mathcal{H}''$.

Step 10) Return $F\mathcal{H}'$.

Theorem 2.17. *It can be decided in time $O(nm^2)$ whether or not a graph G belongs to one of the families: $\mathcal{J}$, $\mathcal{H}'$, $\mathcal{H}''$.*

Proof. Since complexity of Algorithm 2.14 is $O(n^2)$, then it results that the complexity of the step 1) is $O(n^2)$.

A vertex $x \in X$ is simplicial if and only if $\Gamma(x)$ is a clique, but determining if a given subset is a clique can be done in $O(n^2)$. Further, checking every vertex whether it is simplicial executes in $O(n^3)$. So the complexity of the step 2) is $O(n^3)$.

The convex hull of a set $S \subseteq X$ can be computed in $O(|d-conv(S)|m)$ time [4]. Since $|d-conv(S)|$ can reach value n, we obtain that the complexity of the step 8) is $O(nm)$.

The family $\widetilde{\mathcal{P}}_{2,t}(G)$ is obtained by applying the convex hull algorithm to set $X \backslash \{x, y\}$ for all adjacent vertices $x, y \in X$. Since $|d-conv(X \backslash \{x, y\})|$ can reach value n, we obtain that the complexity of the step 3) is $O(nm^2)$.

Clearly, steps 4), 5) and 10) run in constant time, steps 6) and 9) run in $O(n)$ time, but step 7) is executed in $O(n^2)$. As a result, we can decide in $O(nm^2)$ time whether or not a graph G belongs to one of the families: $\mathcal{J}$, $\mathcal{H}'$, $\mathcal{H}''$. □

Theorem 2.18. *Let $G = (X; U) \in \mathcal{H}'$ be a graph that has a convex $(2,t)$-cover $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x, y\}\}$ and has a convex $(2, nt)$-cover. Then G has a convex $(2, nt)$-cover $\mathcal{P}_{2,nt}(G) = \{S_1, S_2\}$ such that exactly one of the following conditions is satisfied:*

a) *$x, y \in S_1$ and $S_2 = X \backslash \{x, y\}$;*

b) *$x \in S_1$, $x \notin S_2$ and $y \in S_2$, $y \notin S_1$.*

Proof. Let $\mathcal{P}'_{2,nt}(G) = \{S'_1, S'_2\}$ be a convex $(2, nt)$-cover of G. Suppose $x, y \in S'_1$. Then, since S_{nt} is nontrivial convex set, we obtain $S_1 = S'_1$ and $S_2 = S_{nt}$. Thus, the condition a) is satisfied. Otherwise the condition b) is satisfied. □

Theorem 2.19. *It can be decided in time $O(n^2m)$ if a graph $G = (X; U) \in \mathcal{H}'$ has a convex $(2, nt)$-cover that satisfies the condition a) of Theorem 2.18. And for this purpose it is sufficient to determine whether there exists $z \in A \cup B$ such that $S_{nt} \nsubseteq d-conv(\{x, y, z\})$, where $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x, y\}\}$ is a convex $(2,t)$-cover of G and $A = \Gamma(x) \backslash \{y\}$, $B = \Gamma(y) \backslash \{x\}$.*

Proof. By definition of $\mathcal{H}'$, G has no simplicial vertices and $|X| \geq 5$. Let $\mathcal{P}^1_{2,nt}(G) = \{S^1_1, S^1_2 = S_{nt}\}$ be a convex $(2, nt)$-cover of G such that $x, y \in S^1_1$.

It is clear that there exists a vertex $z \in A \cup B$ such that the relation $d - conv(\{x, y, z\}) \subseteq S^1_1$ is satisfied. Furthermore, graph G has a convex $(2, nt)$-cover:

$$\mathcal{P}^2_{2,nt}(G) = \{S^2_1 = d - conv(\{x, y, z\}), S^2_2 = S_{nt}\}.$$

Without loss of generality it is sufficient to determine whether there exists $z \in A \cup B$ such that $S_{nt} \nsubseteq d - conv(\{x, y, z\})$. For this purpose we compute the convex hull of $\{x, y, z\}$ for all $z \in A \cup B$. If there is at least one vertex $z \in A \cup B$ such that $S_{nt} \nsubseteq d - conv(\{x, y, z\})$, then G has a convex $(2, nt)$-cover that satisfies the condition a) of Theorem 2.18.

Let us remind that computing of the convex hull of a set $S \subseteq X$ can be done in $O(|d - conv(S)|m)$ time [4]. The decision whether G has a convex $(2, nt)$-cover that satisfies the condition a) of Theorem 2.18 can be obtained by applying the convex hull algorithm at most $|A \cup B|$ times. Thus, the overall complexity is $O(n^2 m)$. □

Theorem 2.20. *Let* $G = (X; U) \in \mathcal{H}'$ *be a graph that has a convex* $(2, t)$*-cover* $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x, y\}\}$ *and has no a convex* $(2, nt)$*-cover that satisfies the condition a) of Theorem 2.18, but has a convex* $(2, nt)$*-cover* $\mathcal{P}_{2,nt}(G) = \{S_1, S_2\}$ *that satisfies the condition b) of Theorem 2.18, that is,* $x \in S_1$*,* $x \notin S_2$ *and* $y \in S_2$*,* $y \notin S_1$*. Then the following conditions are satisfied:*

a) $(\Gamma(x) \backslash y) \subseteq S_1$ *and* $(\Gamma(x) \backslash y) \cap S_2 = \emptyset$*;*
b) $(\Gamma(y) \backslash x) \subseteq S_2$ *and* $(\Gamma(y) \backslash x) \cap S_1 = \emptyset$*.*

Proof. Assume $(\Gamma(x) \backslash y) \cap S_2 \neq \emptyset$, or $(\Gamma(x) \backslash y) \nsubseteq S_1$, i.e., $(\Gamma(x) \backslash y) \cap S_2 \neq \emptyset$. Therefore, we get $x \in S_2$. Since $x \in S_1$ and $y \in S_2$, this means that $\mathcal{P}_{2,nt}(G)$ does not satisfy the condition b) of Theorem 2.18. We have a contradiction. By the same argument, if we assume $(\Gamma(y) \backslash x) \cap S_1 \neq \emptyset$, or $(\Gamma(y) \backslash x) \nsubseteq S_2$, then we also get a contradiction. □

3. NP-completeness

It is known that determining if a graph has a convex 2-cover is NP-complete [7]. Generally, knowing all convex $(2, t)$-covers of a graph G does not facilitate determining if G has a convex $(2, nt)$-cover. But it is useful to know if a graph that has convex $(2, t)$-covers also has a convex $(2, nt)$-cover.

In previous section we proved that all graphs of the families $\mathcal{J}$ and $\mathcal{H}''$ have a convex $(2, nt)$-cover and none graph of $\mathcal{F}$ has a convex $(2, nt)$-cover. Also, we proved that it can be determined in polynomial time whether or not a graph belongs to one of the families: $\mathcal{F}$, $\mathcal{J}$, $\mathcal{H}'$, $\mathcal{H}''$.

Denote by $\mathcal{H}'(2, nt)$ the problem of deciding whether a graph $G \in \mathcal{H}'$ has a convex $(2, nt)$-cover.

Now let us prove that the $\mathcal{H}'(2, nt)$ problem is NP-complete. For this purpose we reduce the NP-complete 1-IN-3 3 SAT problem [5] to the $\mathcal{H}'(2, nt)$ problem.

1-*IN*-3 3 *SAT problem:*

Instance: Set $V = \{v_1, v_2, \ldots, v_n\}$ of variables, collection $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$ of clauses over V such that each clause $c \in \mathcal{C}$ has $|c| = 3$ and no negative literals.

Question: Is there a truth assignment for V such that each clause in $\mathcal{C}$ has exactly one true literal?

We say that $\mathcal{C}$ is *satisfiable* if there exists a truth assignment for V such that $\mathcal{C}$ is satisfiable and each clause in $\mathcal{C}$ has exactly one true variable.

Theorem 3.1. *The $\mathcal{H}'(2, nt)$ problem is NP-complete.*

Proof. $\mathcal{H}'(2, nt)$ problem is in NP, because verifying if a set is convex can be done in polynomial time [4] and nontriviality is verifying in constant time. Further, we reduce 1-IN-3 3 SAT to the $\mathcal{H}'(2, nt)$ problem. First, we determine the structure of a particular graph $G = (X; U) \in \mathcal{H}'$ from a generic instance $(V, \mathcal{C})$ of 1-IN-3 3 SAT. Next, we prove that $\mathcal{C}$ is satisfiable if and only if G has a convex $(2, nt)$-cover. For this purpose we prove that a convex $(2, nt)$-cover of G defines a truth assignment that satisfies $(V, \mathcal{C})$. At the same time, we prove that a truth assignment that satisfies $(V, \mathcal{C})$ defines a convex $(2, nt)$-cover of G.

Let graph G be given by vertex set X and edge set U.

The vertex set X consists of:

a) vertices y and z;
b) $\mathcal{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Y' = \{f, y_5, y_6, y_7, y_8, y_9\}$, $Z = \{z_1, z_2, z_3, z_4\}$, $Z' = \{t, z_5, z_6, z_7, z_8, z_9\}$;
c) $F = \{f_j | 1 \le j \le m\}$, $T = \{t_j | 1 \le j \le m\}$;
d) $L = \{l_j^i | 1 \le j \le m, 1 \le i \le 3\}$, $\mathcal{L} = \{\ell_j^i | 1 \le j \le m, 1 \le i \le 3\}$, $Q = \{q_j^i | 1 \le j \le m, 1 \le i \le 3\}$.

We get $X = \{y, z\} \cup \mathcal{V} \cup Y \cup Y' \cup Z \cup Z' \cup F \cup T \cup L \cup Q \cup \mathcal{L}$. Every variable $v_i \in V$ corresponds to vertex $\mathfrak{v}_i \in \mathcal{V}$. Every clause $c_j \in \mathcal{C}$ corresponds to eleven vertices: f_j, l_j^1, l_j^2, l_j^3, ℓ_j^1, ℓ_j^2, ℓ_j^3, q_j^1, q_j^2, q_j^3, t_j.

The edge set U satisfies the conditions:

a) $y \sim z$, $y_4 \sim z_k$ and $z_4 \sim y_k$ for $1 \le k \le 4$;
b) $\mathcal{V} \cup Q$, $Y \cup \{y\}$ and $Z \cup \{z\}$ are cliques in G;
c) $\Gamma(f) = \mathcal{V} \cup Q \cup F \cup Y \cup \{y_6, y_7\}$ and $\Gamma(t) = \mathcal{V} \cup Q \cup T \cup Z \cup \{z_6, z_7\}$;

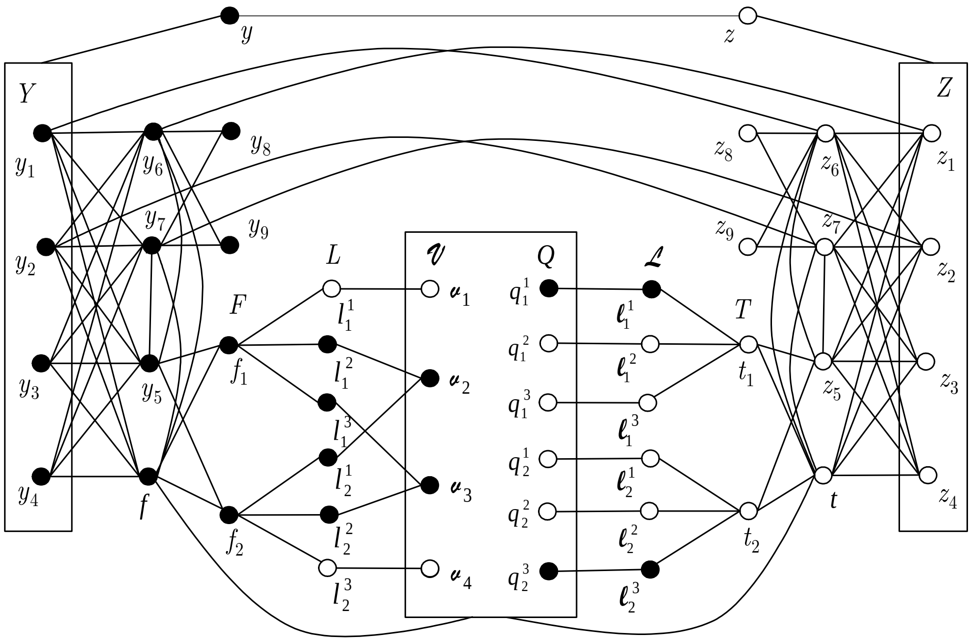

FIGURE 2. The convex $(2, nt)$-cover of the graph G for the instance $(V, \mathcal{C}) = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\})$

d) $\Gamma(y_5) = F \cup Y \cup \{y_6, y_7\}$, $\Gamma(y_6) = Y \cup \{f, y_5, y_8, y_9, z_1\}$, $\Gamma(y_7) = Y \cup \{f, y_5, y_8, y_9, z_2\}$ and $\Gamma(z_5) = T \cup Z \cup \{z_6, z_7\}$, $\Gamma(z_6) = Z \cup \{t, z_5, z_8, z_9, y_1\}$, $\Gamma(z_7) = Z \cup \{t, z_5, z_8, z_9, y_2\}$;

e) every clause $\boldsymbol{c}_j = \{v_a, v_b, v_c\}$, $1 \leq j \leq m$, corresponds to eighteen edges: $\{l_j^1, v_a\}$, $\{l_j^2, v_b\}$, $\{l_j^3, v_c\}$, $\{l_j^1, f_j\}$, $\{l_j^2, f_j\}$, $\{l_j^3, f_j\}$, $\{\boldsymbol{\ell}_j^1, t_j\}$, $\{\boldsymbol{\ell}_j^2, t_j\}$, $\{\boldsymbol{\ell}_j^3, t_j\}$, $\{q_j^1, \boldsymbol{\ell}_j^1\}$, $\{q_j^2, \boldsymbol{\ell}_j^2\}$, $\{q_j^3, \boldsymbol{\ell}_j^3\}$, $\{l_j^1, \boldsymbol{\ell}_j^2\}$, $\{l_j^1, \boldsymbol{\ell}_j^3\}$, $\{l_j^2, \boldsymbol{\ell}_j^1\}$, $\{l_j^2, \boldsymbol{\ell}_j^3\}$, $\{l_j^3, \boldsymbol{\ell}_j^1\}$, $\{l_j^3, \boldsymbol{\ell}_j^2\}$.

We skip the trivial case $|\mathcal{C}| = 1$ of 1-IN-3 3 SAT problem. Consider $|\mathcal{C}| \geq 2$.

Firstly, we show that the obtained graph $G = (X; U)$ belongs to $\mathcal{H}$. Let us remember that $\mathcal{H}$ is a family of simple connected graphs on $n \geq 5$ vertices that have exactly one convex $(2, t)$-cover. According to Theorem 2.3, G has no simplicial vertices. It follows easily from construction of G that this graph really has no such vertices but contains the only one pair of adjacent vertices $\{y, z\}$, which satisfies the conditions of Theorem 2.5. This means that G has exactly the one convex $(2, t)$-cover $\mathcal{P}_{2,t} = \{S_t = \{y, z\}, S_{nt} = X \backslash \{y, z\}\}$ and further G belongs to $\mathcal{H}$.

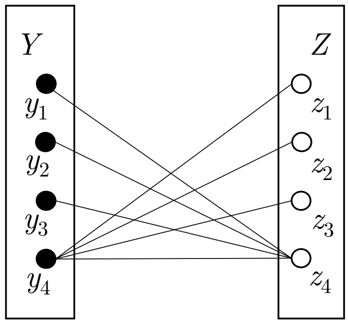

FIGURE 3. Edges between Y and Z

Secondly, we show that G is in $\mathcal{H}'$. To do this we show that all the properties, which characterize the family $\mathcal{H}'$ are satisfied. Clearly, we see that properties a), b) are satisfied. Since $\{y_6, y_7, z_6, z_7\} \subseteq d-conv(A \cup B)$, $d-conv(\{y_6, y_7, z_6, z_7\}) = S_{nt}$ and $\{A \cup B\} \subseteq S_{nt}$, the properties c) and d) are also satisfied. This means that G is in $\mathcal{H}'$.

Thirdly, we show that G has no a convex $(2, nt)$-cover that satisfies the condition a) of Theorem 2.18. By construction of G, $S_{nt} \subseteq d-conv(\{y, z, x\})$ for all $x \in A \cup B$, where $A = \Gamma(y)\backslash\{z\}$ and $B = \Gamma(z)\backslash\{y\}$. Further, taking into account Theorem 2.19, we obtain that G has no a convex $(2, nt)$-cover that satisfies the condition a) of Theorem 2.18. Thus, if graph G has a convex $(2, nt)$-cover, then it satisfies the condition b) of Theorem 2.18 and satisfies Theorem 2.20.

We prove that $\mathcal{C}$ is satisfiable if and only if G has a convex $(2, nt)$-cover.

If $G = (X; U)$ has a convex $(2, nt)$-cover, then $\mathcal{C}$ is satisfiable.

Let $\mathcal{P}_2(G) = \{S_f, S_t\}$ be a convex $(2, nt)$-cover of G such that $y \in S_f$, $y \notin S_t$ and $z \in S_t$, $z \notin S_f$. We have $d-conv(\{y_i, z_j\}) = S_{nt} = X\backslash\{y, z\}$ for every $i, j \in \{8, 9\}$. Further, $y_8, y_9 \in S_f$, $z_8, z_9 \in S_t$ and let $S_1 = Y \cup Y' \cup F$, $S_2 = Z \cup Z' \cup T$.

Let us distinguish some properties:

1) $S_1 \cap S_t = \emptyset$ and $S_2 \cap S_f = \emptyset$.

We see what $S_1 \subseteq d{-}conv(\{y_8, y_9\})$, $S_2 \subseteq d{-}conv(\{z_8, z_9\})$. Consequently we have $S_1 \subseteq S_f$, $S_2 \subseteq S_t$.

Moreover, for each $u \in S_1$, we get $d-conv(\{u, z_8, z_9\}) = S_{nt}$. This implies that $u \notin S_t$ for each $u \in S_1$. Similarly, for each $u \in S_2$, we get

$d - conv(\{u, y_8, y_9\}) = S_{nt}$. This implies that $u \notin S_f$ for each $u \in S_2$. Thus $S_1 \cap S_t = \emptyset$ and $S_2 \cap S_f = \emptyset$.

2) Sets $L, \mathcal{V}, Q, \mathcal{L}$ are uniquely interdependent.

If vertex l_j^i belongs to S_t, then $\Gamma(l_j^i) \cap \mathcal{V} \subseteq S_t$ and ℓ_j^k belongs to S_t for $1 \leq k \leq 3$, $k \neq i$.

If vertex $\boldsymbol{u}_i$ belongs to S_t, then $\Gamma(\boldsymbol{u}_i) \cap L \subseteq S_t$ and for all $l_j^a \in \Gamma(\boldsymbol{u}_i) \cap L$ vertices ℓ_j^k belong to S_t for $1 \leq k \leq 3$, $k \neq a$.

Vertex ℓ_j^i belongs to S_f if and only if q_j^i belongs to S_f. If vertex ℓ_j^i belongs to S_f, then $L' = \{l_j^k | 1 \leq k \leq 3, k \neq i\} \subseteq S_f$ and $\Gamma(l_j^k) \cap \mathcal{V}$ is contained in S_f for all $l_j^k \in L'$.

3) Exactly one vertex of $L_j = \{l_j^1, l_j^2, l_j^3\}$ belongs to S_t, for $1 \leq j \leq m$, and exactly one vertex of $\mathcal{L}_j = \{\ell_j^1, \ell_j^2, \ell_j^3\}$ belongs to S_f, for $1 \leq j \leq m$.

Exactly one vertex of every set $L_j = \{l_j^1, l_j^2, l_j^3\}$, $1 \leq j \leq m$, belongs to S_t. In the converse case, if two vertices $\{l_j^a, l_j^b\}$ of L_j belong to S_t, then f_j belongs to S_t. By Property 1, we get a contradiction. If none vertex of $L_j = \{l_j^1, l_j^2, l_j^3\}$ belongs to S_t, then $L_j \subseteq S_f$, $\mathcal{L}_j = \{\ell_j^1, \ell_j^2, \ell_j^3\} \subseteq S_f$ and t_j belongs to S_f. Now by Property 1, we have a contradiction.

In addition, exactly one vertex of every set $\mathcal{L}_j = \{\ell_j^1, \ell_j^2, \ell_j^3\}$, $1 \leq j \leq m$, belongs to S_f.

We associate $\mathcal{V}$ with V and L with $\mathcal{C}$ such that convex $(2, nt)$-cover represents a truth assignment for $\mathcal{V}$, where the variable v_i is true if and only if the vertex $\boldsymbol{u}_i \in S_t$.

Let us remark that sets S_f, S_t are nontrivial and disjoint. It follows from Properties 1 - 3 that if G has a convex $(2, nt)$-cover $\mathcal{P}_2(G) = \{S_f, S_t\}$, then $\mathcal{C}$ is satisfiable.

If $\mathcal{C}$ is satisfiable, then $G = (X; U)$ has a convex $(2, nt)$-cover.

Suppose that there exists a truth assignment, which satisfies $(V, \mathcal{C})$. We construct a convex $(2, nt)$-cover $\mathcal{P}_2(G) = \{S_f, S_t\}$ as follows:

Step 1. Define $S_t = Z \cup Z' \cup T \cup \{z\}$;

Step 2. For each true variable v_i of V we add vertex $\boldsymbol{u}_i$ and the set $L' = \Gamma(\boldsymbol{u}_i) \cap L$ to S_t and for each $l_j^a \in L'$ we add vertices q_j^b, ℓ_j^b to S_t such that $\ell_j^b \sim l_j^a$ and $q_j^b \sim \ell_j^b$;

Step 3. Define $S_f = X \backslash S_t$.

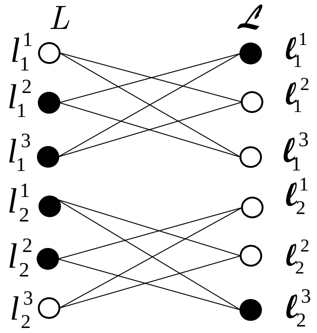

FIGURE 4. Edges between L and $\mathcal{L}$

Clearly, for the resulting convex $(2, nt)$-cover $\mathcal{P}_2(G) = \{S_f, S_t\}$ the Properties 1, 2 and 3 are satisfied. Note also that sets S_f and S_t are disjoint. Hence, if $\mathcal{C}$ is satisfiable, then G has a convex $(2, nt)$-cover.

We represent in Figure 2 the graph G that corresponds to a particular instance $(V, \mathcal{C}) = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\})$. Sets $Q \cup \mathcal{V} \cup \{f\}$, $Q \cup \mathcal{V} \cup \{t\}$, $Y \cup \{y\}$ and $Z \cup \{z\}$ generate cliques in G. White vertices belong to S_t and black vertices belong to S_f. White vertices of $\mathcal{V}$ represent the variables of V set to true. All edges between Y and Z are represented in Figure 3 but all edges between L and $\mathcal{L}$ are represented in Figure 4. □

Finally, we obtain that it is NP-complete do decide whether a graph that has convex $(2, t)$-covers also has a convex $(2, nt)$-cover. Indeed, this follows from fact that the $\mathcal{H}'(2, nt)$ problem is NP-complete.

4. Some graph classes, which have a convex $(2, nt)$-cover

Let us examine some classes of simple connected graphs, which have a convex $(2, nt)$-cover.

Consider C_n a cycle graph on n vertices. Recall that a *chordal* graph is a connected graph such that every cycle of length at least 4 has a chord.

Theorem 4.1. *A chordal graph G on $n \geq 4$ vertices has a convex $(2, nt)$-cover.*

Proof. Every chordal graph G contains at least one simplicial vertex [6]. Also, every chordal graph on $n = 4$ vertices is not equal to the cycle C_4. This yields that under the conditions of Corollary 2.2 and Theorem 2.4, chordal graph G on $n \geq 4$ vertices has a convex $(2, nt)$-cover. □

Corollary 4.2. *A tree and a complete graph on $n \geq 4$ vertices have a convex $(2, nt)$-cover.*

Corollary 4.2 follows directly from the fact that these types of graphs are subclusses of chordal graphs.

A *power of cycle* C_n^k, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, is a graph such that $X(C_n^k) = X(C_n)$ and $U(C_n^k) = \{\{u_i, u_j\} | u_i, u_j \in X(C_n^k),\ d_{C_n}(u_i, u_j) \leq k\}$.

In [3] it is established the following theorem, which states conditions to determine whether C_n^k has a convex 2-partition.

Theorem 4.3. [3] *C_n^k has a convex 2-partition if and only if $n \leq 2k+2$ or $n \equiv 0, 1, 2$ (mod $2k$).*

Using Theorem 4.3, we have the following result.

Theorem 4.4. *C_n^k has a convex $(2, nt)$-cover if and only if $n \geq 4$, $C_n^k \neq C_4$, and $n \leq 2k+2$ or $n \equiv 0, 1, 2$ (mod $2k$).*

Proof. First, we shall show that C_n^k has a convex 2-partition if and only if C_n^k has a convex 2-cover. By construction of C_n^k, every convex set of C_n^k consists of consecutive vertices of C_n. Suppose $\boldsymbol{\mathcal{P}}_2(C_n^k) = \{S_1, S_2\}$ is a convex 2-cover of C_n^k. Subtracting $S_1 \cap S_2$ from S_1 or from S_2, we get a convex 2-partition of C_n^k. Therefore, every convex 2-cover of C_n^k can be transformed in a convex 2-partition. Recall that convex 2-partition is a convex 2-cover.

Let us show that C_n^k has a convex 2-cover if and only if C_n^k has a convex $(2, nt)$-cover and conditions $n \geq 4$, $C_n^k \neq C_4$ hold.

For $n \leq 3$ there is no convex $(2, nt)$-cover of graph C_n^k. It remains to verify if C_n^k has a convex $(2, nt)$-cover for $n \geq 4$.

Assume that $n = 4$. According to power of cycle definition, we have $1 \leq k \leq 2$. If $k = 1$, then $C_4^1 = C_4$. By Corollary 2.2, it follows that this graph has no a convex $(2, nt)$-cover. On the other hand, if $k = 2$, then $C_4^2 = K_4$ and the application of Corollary 4.2 yields that C_4^2 has a convex $(2, nt)$-cover.

Further, assume that $n \geq 5$. Suppose $\boldsymbol{\mathcal{P}}_{2,t}(C_n^k) = \{S_t, S_{nt}\}$ is a convex $(2, t)$-cover. If $|S_t| = 1$, or if $|S_t| = 2$ and $S_t \cap S_{nt} \neq \emptyset$, then taking into account Theorem 2.3 and Theorem 2.4, C_n^k has a convex $(2, nt)$-cover. Otherwise if $|S_t| = 2$ and $S_t \cap S_{nt} = \emptyset$, then since the construction of power of cycle is regular, graph C_n^k has the another convex $(2, t)$-cover $\boldsymbol{\mathcal{P}}'_{2,t}(C_n^k) = \{S'_t, S'_{nt}\}$ such that S'_t consists of two consecutive vertices in C_n and $S_t \cap S'_t = \emptyset$, where $S'_t \subset S_{nt}$ and $S_t \subset S'_{nt}$. Thus, using Theorem 2.6, we get a convex $(2, nt)$-cover of C_n^k. □

A cactus graph is a connected graph in which any two graph cycles have at most one vertex in common.

Theorem 4.5. *A cactus graph G on n vertices has a convex $(2, nt)$-cover if and only if $n \geq 4$, $G \neq C_4$.*

Proof. Using Corollary 2.2, we know that a connected graph on 4 vertices has a convex $(2, nt)$-cover if and only if this graph is different from C_4. This implies that a cactus graph G on 4 vertices also has a convex $(2, nt)$-cover if and only if G is different from C_4.

Suppose $n \geq 5$. If G contains a simplicial vertex, then taking into account Theorem 2.4, graph G has a convex $(2, nt)$-cover. Assume that G has no simplicial vertices. If G is a cycle $C_n = C_n^1$, then by Theorem 4.4 graph G has a convex $(2, nt)$-cover. Otherwise G has a cut vertex v that is adjacent to $k \geq 2$ various connected components $S_1, S_2, \ldots, S_k$. Further, since G has no simplicial vertices, we have $|X(S_i)| \geq 2$ for $1 \leq i \leq k$. Thus, graph G has a convex $(2, nt)$-cover: $\mathcal{P}_{2,nt}(G) = \{\{v\} \cup \bigcup_{1 \leq i \leq k-1} X(S_i), X(S_k) \cup \{v\}$. □

5. Conclusion

The paper is a continuation of computational complexity research of convex two cover problem, declared open in [2]. We proved NP-completness of this problem in [7]. In the article we establish the existence of a convex $(2, nt)$-cover in dependency on existing convex $(2, t)$-covers. Generally, we prove that it is NP-complete do decide whether a graph that has convex $(2, t)$-covers also has a convex $(2, nt)$-cover. Finally, we show that some graphs on $n \geq 4$ vertices implicitly have a convex $(2, nt)$-cover. In particular, chordal graphs and cactus graphs, different from C_4, are covered by two nontrivial convex sets.

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