STUDIA UNIV. BABEŞ–BOLYAI, INFORMATICA, Volume **LX**, Number 2, 2015

AN INFEASIBLE INTERIOR-POINT METHOD FOR THE CARTESIAN $P_*(\kappa)$ SECOND-ORDER CONE LINEAR COMPLEMENTARITY PROBLEM WITH ONE CENTERING STEP

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Abstract. In this paper, we present a new full step infeasible interior-point algorithm for the Cartesian $P_*(\kappa)$ linear complementarity problem over second-order cones. The algorithm uses only full Nesterov and Todd steps. Each (main) iteration of the algorithm consists of one so-called feasibility step and only one centering step. The algorithm starts with a strictly feasible point of a perturbed problem, after an iteration, the new iterate is still strictly feasible of the new perturbed problem. The algorithm has the same complexity as the best known infeasible interior-point methods.

1. Introduction

In this paper, we consider the second-order cone linear complementarity problem (SOCLCP), which seeks vectors $x, s \in R^n$ such that

$$x \in \mathcal{K},\ s \in \mathcal{K},\ s = \mathcal{A}(x) + q,\ \langle x, s\rangle = 0,$$

where $\langle x, s\rangle := \mathrm{tr}(x \circ s)$ denotes the Euclidean inner product, $q \in R^n$, $\mathcal{A} : \mathcal{K} \to \mathcal{K}$ is a linear transformation, and $\mathcal{K} \subseteq R^n$ is the Cartesian product of several second-order cones, i.e., $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2 \times \cdots \times \mathcal{K}^N$, with

$$\mathcal{K}^j := \left\{(x_1, x_{2:n_j}^T)^T \in R \times R^{n_j-1} : x_1 \geq \|x_{2:n_j}\|\right\},\ \text{where } x_{2:n_j} := (x_2; \ldots; x_{n_j})$$

for each $j = 1, \ldots, N$ and $\sum_{j=1}^{N} n_j = n$. Since $\mathcal{K}$ has finite dimensional, we can consider matrix representation of the linear transformation $\mathcal{A}(x) = Mx$,

Received by the editors: March 4, 2015.

2010 *Mathematics Subject Classification.* 90C51, 90C33.

1998 *CR Categories and Descriptors.* G.1.6. [**Mathematics of Computing**]: Numerical Analysis – Optimization – *Nonlinear Programming.*

Key words and phrases. The Cartesian product of second-order cones, Linear complementarity problem, Infeasible interior-point method, Polynomial complexity.

with $M \in R^{n\times n}$. By Lemma 2.2 in [7] we know that $\langle x, s\rangle = 0$ if and only if $x \circ s = 0$. Therefore, we may rewrite SOCLCP in the following form

$$x \in \mathcal{K},\ s \in \mathcal{K},\ s = Mx + q,\ x \circ s = 0. \tag{1}$$

We call SOCLCP the Cartesian $P_*(\kappa)$-SOCLCP if the matrix M has the Cartesian $P_*(\kappa)$-property, i.e., for any $\kappa \geq 0$, the matrix M satisfies

$$\langle x, Mx\rangle \geq -4\kappa \sum_{j\in I_+(x)} \langle x^{(j)}, [Mx]^{(j)}\rangle, \text{ where } I_+(x) = \{j : \langle x^{(j)}[Mx]^{(j)}\rangle \geq 0\}.$$

The concept of the Cartesian $P_*(\kappa)$-property was first introduced by Luo and Xiu [15] in the general Euclidean Jordan algebra. Actually, it is a straightforward extension of the $P_*(\kappa)$-matrix introduced by Kojima et al. [14]. Moreover, the matrix M with the Cartesian $P_*(\kappa)$-property becomes the usual $P_*(\kappa)$-matrix when $\mathcal{K}$ is specified to be R^n_+, correspondingly, the Cartesian $P_*(\kappa)$-SOCLCP reduces to the $P_*(\kappa)$-LCP [15]. Wang and Zhu [25] presented a primal-dual interior-point algorithm for the Cartesian $P_*(\kappa)$-SOCLCP based on a parametric kernel function. The primal-dual full-Newton step feasible IPM for linear optimization (LO) was first analyzed by Roos et al. [18]. Darvay [4] proposed a full-Newton step primal-dual path-following interior-point algorithm for LO which is based on the equivalent algebraic transformation. Achache [1], Wang and Bai [22, 23] and Wang [24] generalized the results for LO in [4] to convex quadratic optimization (CQO), second-order cone optimization (SOCO), symmetric cone optimization (SCO) and monotone LCP over symmetric cone (SCLCP).

The above algorithms enjoy the best known iteration bound. However, they are all feasible IPMs, which start with a strictly feasible interior point and maintain feasibility during the solution process. One may distinguish between feasible IPMs and infeasible IPMs (IIPMs), which start with an arbitrary positive point and feasibility is reached as optimality is approached. In 2006, Roos [17] designed the first full-Newton step primal-dual IIPM with the currently best iteration bound for LO. Following Roos' contribution, Kheirfam and Mahdavi-Amiri [11] and Gu et al. [8] respectively extended both versions of the feasible IPM [18] and IIPM [17] to SCLCP and SCO by using Nesterov and Todd (NT) direction as a search direction and obtained the same iteration complexity bounds. Kheirfam and Mahdavi-Amiri [12] presented a full NT-step IIPM for SCLCP based on modified NT directions, and the corresponding complexity results accord with the currently best-known iteration bound for IIPMs. Based on Darvay's technique [4] extension to SCO in [23], Kheirfam [10] presented a full-NT step IIPM for SCO. Recently, Kheirfam [9] designed and analyzed the full-Newton step IIPM based on a new proximity measure for $P_*(\kappa)$ horizontal linear complementarity problem (HLCP). All

IIPMs mentioned so far consists of one feasibility step and a few - at most three - centering steps. Recently, Darvay et al. [5] presented an improved version of an IIPM for LO in [2], in sense that each iteration of the algorithm consists of one feasibility step and only a centering step.

Motivated by Darvay et al.'s recent work, we present a new full NT-step IIPM for the Cartesian $P_*(\kappa)$-SOCLCP based on the technique introduced in [5] and prove that each main iteration needs to a feasibility step and one centering step in order to get a well-defined algorithm. The new algorithm reduces the searching steps in each iteration and tendering an interesting analysis for iteration complexity.

The remainder of our work is organized as follows. In Section 2, we briefly recall the corresponding Euclidean Jordan algebra to second-order cones. Based on Darvay's technique, we are providing some new results that will be used in the complexity analysis of the algorithm. In Section 3, we introduce the perturbed problem and the new infeasible interior-point algorithm. Then, we provide the complexity analysis of the algorithm and derive the iteration bound. Finally, some conclusions are given in Section 4.

2. Euclidean Jordan algebra and some results

In this section, we first recall some basic concepts of Euclidean Jordan algebra [3, 6], and then we provide some results that will be used for the main purpose of this paper.

A Euclidean Jordan algebra $(\mathcal{J}, \langle\cdot,\cdot\rangle, \circ)$ ($\mathcal{J}$ for short) is an n-dimensional inner product space over R endowed with a bilinear map $\circ : \mathcal{J}\times\mathcal{J} \to \mathcal{J}$ iff for all $x, y, z \in \mathcal{J}, x\circ y = y\circ x, x\circ(x^2\circ y) = x^2\circ(x\circ y)$ and $\langle x\circ y, z\rangle = \langle x, y\circ z\rangle$ where $x^2 := x\circ x$. For any $x^j = (x_1^j, x_{2:n_j}^j), s^j = (s_1^j, s_{2:n_j}^j) \in R\times R^{n_j-1}$, the Jordan product of x^j and s^j is defined as

$$x^j\circ s^j = \big((x^j)^T s^j; x_1^j s_{2:n_j}^j + s_1^j x_{2:n_j}^j\big).$$

One can easily verify that $(R^{n_j}, \circ)$ is a Euclidean Jordan algebra, with $e^j = (1; 0) \in R\times R^{n_j-1}$ as identity element. In this algebra, the second-order cone $\mathcal{K}^j$ is the cone of square, i.e., $\mathcal{K}^j = \{x^2 : x \in R^{n_j}\}$ (see, [6]). Given a vector $x^j = (x_1^j, x_{2:n_j}^j) \in R\times R^{n_j-1}$, let

$$L(x^j) := \begin{bmatrix} x_1^j & (x^j)_{2:n_j}^T \\ x_{2:n_j}^j & x_1^j E_{n_j-1} \end{bmatrix},$$

which can be viewed as a linear mapping from R^{n_j-1} to R^{n_j-1}, where E_{n_j-1} denotes the identify matrix. It is not hard to verify that $L(x^j)s^j = x^j\circ s^j$

for any $x^j, s^j \in R^{n_j}$. The eigenvalues of $L(x^j)$ are denoted respectively as $\lambda_{\min}(x^j) = x_1^j - \|x_{2:n_j}^j\|$ and $\lambda_{\max}(x^j) = x_1^j + \|x_{2:n_j}^j\|$. Note that

$$x^j \in \mathcal{K}^j \Leftrightarrow \lambda_{\min}(x^j) \geq 0,\ x^j \in \operatorname{int}\mathcal{K}^j \Leftrightarrow \lambda_{\min}(x^j) > 0,$$

where $\operatorname{int}\mathcal{K}^j$ denotes the interior of $\mathcal{K}^j$. For any $x^j \in R^{n_j}, P(x^j) := 2L(x^j)^2 - L((x^j)^2)$ where $L(x^j)^2 = L(x^j)L(x^j)$. The map $P(x^j)$ is called the quadratic representation of x^j. Each $x^j = (x_1^j; x_{2:n_j}^j) \in R^{n_j}$ admits a spectral decomposition, associated with $\mathcal{K}^j$, of the form $x^j = \lambda_{\max}(x^j)c_1 + \lambda_{\min}(x^j)c_2$, where c_1, c_2 are the associated eigenvectors given by

$$c_1 = \frac{1}{2}\big(1; \frac{x_{2:n_j}^j}{\|x_{2:n_j}^j\|}\big),\ c_2 = \frac{1}{2}\big(1; \frac{-x_{2:n_j}^j}{\|x_{2:n_j}^j\|}\big). \tag{2}$$

Moreover, $\operatorname{tr}(x^j) = \lambda_{\max}(x^j) + \lambda_{\min}(x^j) = 2x_1^j$. The natural inner product is given by

$$\langle x^j, s^j\rangle := \operatorname{tr}(x^j \circ s^j) = 2(x^j)^T s^j,\ x^j, s^j \in R^{n_j}.$$

Hence, the norm induced by this inner product, which is denoted by $\|\cdot\|_F$, satisfies

$$\|x^j\|_F = \sqrt{\langle x^j, x^j\rangle} = \sqrt{\operatorname{tr}((x^j)^2)} = \sqrt{\lambda_{\min}(x^j)^2 + \lambda_{\max}(x^j)^2} = \sqrt{2}\|x^j\|.$$

In the sequel, we generalize the above definitions and properties to the case where $N > 1$, when the second-order cone underlying $\mathcal{K}$ is the Cartesian product of N second-order cones $\mathcal{K}^j$. For any $x = (x^1; \cdots; x^N) \in R^n$ with $x^j \in R^{n_j}, j = 1, \ldots, N$, the algebra $(R^n, \circ)$ is defined as a direct product of the Jordan algebras $(R^{n_j}, \circ)$ as

$$x \circ s := (x^1 \circ s^1; \cdots; x^N \circ s^N).$$

Obviously, if $e^j \in \mathcal{K}^j$ is the identity element in the Jordan algebra for the jth second-order cone, then the vector $e = (e^1; \cdots; e^N)$ is the identity element in $(R^n, \circ)$. Moreover, $\operatorname{tr}(e) = 2N$, which is the rank of $(R^n, \circ)$. The matrix $L(x)$ and the quadratic representation $P(x)$ of $(R^n, \circ)$ can be respectively adjusted to

$$L(x) := \operatorname{diag}\big(L(x^1), \cdots, L(x^N)\big),\ P(x) := \operatorname{diag}\big(P(x^1), \cdots, P(x^N)\big).$$

Furthermore

$$\lambda_{\max}(x) = \max_{1\leq j\leq N}\{\lambda_{\max}(x^j)\},\ \lambda_{\min}(x) = \min_{1\leq j\leq N}\{\lambda_{\min}(x^j)\}, \|x\|_F^2 = \sum_{j=1}^{N}\|x^j\|_F^2$$

$$\operatorname{tr}(x) = \sum_{j=1}^{N}\operatorname{tr}(x^j) = \sum_{j=1}^{N}\big(\lambda_{\max}(x^j) + \lambda_{\min}(x^j)\big) := \sum_{j=1}^{2N}\lambda_j(x).$$

Lemma 2.1. (Corollary 2.14 in [22]) *Let* $x, s \in R^n$ *and* $x+s=e$. *If* $|\lambda_{\min}(s)|$ *is small enough, then* $\psi(x+s) \approx \psi(x)+\psi^{'}(x)\circ s$, *where* $\psi(t):[0,\infty)\to(0,\infty)$ *such that* $\psi^{'}(t)>0$ *for all* $t>0$.

Lemma 2.2. (Lemma 6.1 in [22]) *Let* $x(\alpha):=x+\alpha\Delta x$ *and* $s(\alpha):=s+\alpha\Delta s$ *for all* $0\leq\alpha\leq 1$. *Suppose that* $x,s\in \mathrm{int}\mathcal{K}$. *If one has*

$$\det\big(x(\alpha)\circ s(\alpha)\big)>0,\ \forall 0\leq\alpha\leq\bar{\alpha},$$

then $x(\bar{\alpha}), s(\bar{\alpha})\in\mathrm{int}\mathcal{K}$.

Lemma 2.3. (Theorem 4 in [20]) *Let* $x, s\in\mathcal{K}$. *Then*

$$\lambda_{\min}\big(P(x)^{\frac{1}{2}}s\big)\geq\lambda_{\min}(x\circ s).$$

Lemma 2.4. (Lemma 30 in [19]) *Let* $x, s\in\mathcal{K}$. *Then*

$$\big\|P(x)^{\frac{1}{2}}s-e\big\|_F\leq\big\|x\circ s-e\big\|_F.$$

Luo and Xiu [15] have discussed the existence and uniqueness of the central path of the Cartesian $P_*(\kappa)$ symmetric cone linear complementarity problem ($P_*(\kappa)$-SCLCP). As a special case of the Cartesian $P_*(\kappa)$-SCLCP, the existence and uniqueness of the central path of the Cartesian $P_*(\kappa)$-SOCLCP could be similarly obtained. The main idea of IPMs is to replace the last equation in (1), the so-called complementarity condition, with the parameterized equation $x\circ s=\mu e$, with parameter $\mu>0$. So we consider the following system

$$s=Mx+q,\ x\circ s=\mu e,\ x,s\in\mathrm{int}\mathcal{K}. \tag{3}$$

Throughout the paper, we assume that the Cartesian $P_*(\kappa)$-SOCLCP satisfies the interior-point condition (IPC), i.e., there exists $x^0, s^0\in\mathrm{int}\mathcal{K}$ with $s^0=Mx^0+q$, then the system (3) has a unique solution $(x(\mu), s(\mu))$, for each $\mu>0$ as the μ-center of the Cartesian $P_*(\kappa)$-SOCLCP. The set of μ-centers is called the central path of the Cartesian $P_*(\kappa)$-SOCLCP. If $\mu\longrightarrow 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields a solution for the Cartesian $P_*(\kappa)$-SOCLCP [26]. Similarly to the LO case [4], we replace the standard centering equation $x\circ s=\mu e$ by $\psi(\frac{x\circ s}{\mu})=\psi(e)$, where $\psi(\cdot)$ is the vector-valued function induced by the univariate function $\psi(t)$. Then, we consider the following system

$$s=Mx+q,\ \psi\Big(\frac{x\circ s}{\mu}\Big)=\psi(e),\ x,s\in\mathrm{int}\mathcal{K}, \tag{4}$$

Applying Newton's method to the system (4) leads to the following system

$$M\Delta x-\Delta s=0,\ \psi\Big(\tfrac{x\circ s}{\mu}+\tfrac{x\circ\Delta s+\Delta x\circ s+\Delta x\circ\Delta s}{\mu}\Big)=\psi(e). \tag{5}$$

Neglecting the term $\Delta x \circ \Delta s$, from Lemma 2.1, we can replace the second equation of (5) by

$$\psi\big(\frac{x \circ s}{\mu}\big) + \psi'\big(\frac{x \circ s}{\mu}\big) \circ \big(\frac{x \circ \Delta s + \Delta x \circ s}{\mu}\big) = \psi(e).$$

This enables us to rewrite the system (5) as follows

$$(6)\quad M\Delta x - \Delta s = 0,\ x \circ \Delta s + s \circ \Delta x = \mu\big(\psi'(\tfrac{x \circ s}{\mu})\big)^{-1}\big(\psi(e) - \psi(\tfrac{x \circ s}{\mu})\big).$$

Due to the fact that x and s do not operator commute in general, i.e., $L(x)L(s) \neq L(s)L(x)$, the above system does not always have a unique solution. To overcome this difficulty, the second equation of the system (4) is replaced by the following equivalent scaled equation (cf. Lemma 28 in [19])

$$\psi\Big(\frac{P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}s}{\mu}\Big) = \psi(e),$$

where $w = P(x)^{\frac{1}{2}}\Big(P(x)^{\frac{1}{2}}s\Big)^{-\frac{1}{2}}\Big[= P(s)^{-\frac{1}{2}}\Big(P(s)^{\frac{1}{2}}x\Big)^{\frac{1}{2}}\Big]$ is the NT-scaling point of x and s. This scaling point was first proposed by Nesterov and Todd for self-scaled cones [16]. Now, we replace the second equation of the system (5) by

$$\psi\Big(\frac{P(w)^{-\frac{1}{2}}(x + \Delta x) \circ P(w)^{\frac{1}{2}}(s + \Delta s)}{\mu}\Big) = \psi(e).$$

Applying Newton's method again and neglecting the term $P(w)^{-\frac{1}{2}}\Delta x \circ P(w)^{\frac{1}{2}}\Delta s$, from Lemma 2.1, we get

$$(7)\qquad \begin{array}{c} M\Delta x - \Delta s = 0, \\ P(w)^{\frac{1}{2}}s \circ P(w)^{-\frac{1}{2}}\Delta x + P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}\Delta s = \\ \mu\big(\psi'(\frac{P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}s}{\mu})\big)^{-1}\big(\psi(e) - \psi(\frac{P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}s}{\mu})\big). \end{array}$$

In this case, assuming that $\psi(t) = \sqrt{t}$, the system (7) becomes

$$(8)\qquad \begin{array}{c} \Delta s - M\Delta x = 0, \\ P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}\Delta s + P(w)^{\frac{1}{2}}s \circ P(w)^{-\frac{1}{2}}\Delta x = \\ 2\Big(\big(\mu P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}s\big)^{\frac{1}{2}} - P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}s\Big). \end{array}$$

We use the following notations:

$$(9)\qquad v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}}\ \Big[= \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}}\Big],\ d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}},\ d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}.$$

It follows from (9) that the system (8) reduces to

$$(10)\qquad \overline{M}d_x - d_s = 0,\ d_x + d_s = p_v,$$

where $\overline{M} := P(w)^{\frac{1}{2}} M P(w)^{\frac{1}{2}}$ and $p_v := 2(e - v)$. The new search directions d_x and d_s are obtained by solving (10) so that Δx and Δs are computed via (9). The new iterates are given by

$$\tilde{x} := x + \Delta x, \;\; \tilde{s} := s + \Delta s. \tag{11}$$

For the analysis of the algorithm, we define a norm-based proximity measure

$$\delta(v) := \delta(x, s; \mu) := \frac{\|p_v\|_F}{2} = \|e - v\|_F. \tag{12}$$

Defining $q_v = d_x - d_s$, we have

$$d_x = \frac{p_v + q_v}{2}, \;\; d_s = \frac{p_v - q_v}{2}, \;\; d_x \circ d_s = \frac{p_v \circ p_v - q_v \circ q_v}{4}. \tag{13}$$

Moreover, since $\overline{M}$ has the Cartesian $P_*(\kappa)$-property and $d_s = \overline{M} d_x$ from the first equation in (10), we obtain

$$\begin{aligned}
\langle d_x, d_s \rangle &\geq -4\kappa \sum_{j \in I_+} \langle d_x^{(j)}, d_s^{(j)} \rangle \geq -\kappa \sum_{j \in I_+} \langle d_x^{(j)} + d_s^{(j)}, d_x^{(j)} + d_s^{(j)} \rangle \\
&\geq -\kappa \sum_{j=1}^{N} \langle d_x^{(j)} + d_s^{(j)}, d_x^{(j)} + d_s^{(j)} \rangle = -\kappa \sum_{j=1}^{N} \big\| d_x^{(j)} + d_s^{(j)} \big\|_F^2 \\
&= -\kappa \big\| d_x + d_s \big\|_F^2 = -\kappa \big\| p_v \big\|_F^2 = -4\kappa\delta^2. \qquad (14)
\end{aligned}$$

This implies that

$$\|q_v\|_F^2 = \|p_v\|_F^2 - 4\langle d_x, d_s \rangle \leq 4\delta^2 + 16\kappa\delta^2 = 4(1 + 4\kappa)\delta^2. \tag{15}$$

Using (9) and (11), we obtain

$$\tilde{x} = x + \Delta x = \sqrt{\mu} P(w)^{\frac{1}{2}} (v + d_x), \;\; \tilde{s} = s + \Delta s = \sqrt{\mu} P(w)^{-\frac{1}{2}} (v + d_s). \tag{16}$$

Since $P(w)^{\frac{1}{2}}$ and its inverse $P(w)^{-\frac{1}{2}}$ are automorphisms of $\mathcal{K}$, then $\tilde{x}$ and $\tilde{s}$ belong to $\mathrm{int}\mathcal{K}$ if and only if $v + d_x$ and $v + d_s$ belong to $\mathrm{int}\mathcal{K}$, respectively.

Lemma 2.5. *Let $\delta(x, s; \mu) < \frac{1}{\sqrt{1+4\kappa}}$. Then $\tilde{x}$ and $\tilde{s}$ are strictly feasible.*

Proof. Using (15) and an argument similar to that described in the proof of lemma 4.2 [23], the result follows. □

According to (9), the v-vector after the step is given by

$$\tilde{v} := \frac{P((\tilde{w})^{-\frac{1}{2}})\tilde{x}}{\sqrt{\mu}} \;\; \Big[= \frac{P((\tilde{w})^{\frac{1}{2}})\tilde{s}}{\sqrt{\mu}} \Big],$$

where $\tilde{w}$ is the NT-scaling point of $\tilde{x}$ and $\tilde{s}$.

Lemma 2.6. (Proposition 5.9.3 in [21]) *One has $\tilde{v} \sim \big(P(v + d_x)^{\frac{1}{2}} (v + d_s)\big)^{\frac{1}{2}}$.*

Lemma 2.7. *Let* $\delta := \delta(x, s; \mu)$. *Then* $\lambda_{\min}(\tilde{v}) \geq \sqrt{1-(1+4\kappa)\delta^2}$.

Proof. From Lemma 2.6, Lemma 2.3 and (15) it follows that

$$\begin{aligned}\lambda_{\min}(\tilde{v}) &= \lambda_{\min}\big((P(v+d_x)^{\frac{1}{2}}(v+d_s)\big)^{\frac{1}{2}}\big) \geq \Big(\lambda_{\min}\big((v+d_x)\circ(v+d_s)\big)\Big)^{\frac{1}{2}}\\ &= \Big(\lambda_{\min}\big(e-\frac{q_v\circ q_v}{4}\big)\Big)^{\frac{1}{2}} \geq \Big(1-\|\frac{q_v\circ q_v}{4}\|_F\Big)^{\frac{1}{2}} \geq \Big(1-\frac{\|q_v\|_F^2}{4}\Big)^{\frac{1}{2}}\\ &\geq \sqrt{1-(1+4\kappa)\delta^2}.\end{aligned}$$

This completes the proof. □

Lemma 2.8. *Let* $\delta := \delta(x, s; \mu) < \frac{1}{\sqrt{1+4\kappa}}$. *Then*

$$\delta(\tilde{x}, \tilde{s}; \mu) \leq \frac{(1+4\kappa)\delta^2}{1+\sqrt{1-(1+4\kappa)\delta^2}}.$$

Proof. Using Lemma 2.7, (15) and an argument similar to that described in the proof of lemma 4.4 [23], the result follows. □

3. Full NT-step IIPM

3.1. **The perturbed problem.** As usually of IIPMs, we assume that the Cartesian $P_*(\kappa)$-SOCLCP (1) has a solution (x^*, s^*) such that

$$\|x^*\|_\infty \leq \rho_p,\ \|s^*\|_\infty \leq \rho_d, \tag{17}$$

where ρ_p and ρ_d are positive. Furthermore, we define

$$x^0 = \rho_p e,\ s^0 = \rho_d e,\ \mu^0 = \rho_p\rho_d, \tag{18}$$

as the initial starting point. Then, the initial residual as is given $r_q^0 = s^0 - Mx^0 - q$. For any ν with $0 < \nu \leq 1$, we consider the perturbed problem to be

$$s - Mx - q = \nu r_q^0,\ x, s \in \mathcal{K}. \tag{19}$$

Note that if $\nu = 1$, then $(x, s) = (x^0, s^0)$ yields a strictly feasible solution of (19). We conclude that if $\nu = 1$, then (19) satisfies the IPC. More generally, we have the following result.

Lemma 3.1. *Let the Cartesian* $P_*(\kappa)$*-SOCLCP be feasible and* $0 < \nu \leq 1$. *Then, the perturbed problem (19) satisfies the IPC.*

Proof. The proof is similar to the proof of Lemma 17 in [11]. □

Let the Cartesian $P_*(\kappa)$-SOCLCP be feasible and $0 < \nu \leq 1$. Lemma 3.1 implies that the perturbed problem (19) satisfies the IPC, for each $0 < \nu \leq 1$, and hence its central path exists. This means that the system

$$s - Mx - q = \nu r_q^0,\ x \circ s = \mu e,\ x, s \in \mathcal{K}, \tag{20}$$

has a unique solution, for every $\mu > 0$. It is the μ-center of the perturbed problem (19). In the sequel, the parameters μ and ν always satisfy the relation $\mu = \mu^0\nu$. The system (20), can be written as follows:

$$s - Mx - q = \nu r_q^0,\ \psi(\frac{x \circ s}{\mu}) = \psi(e),\ x, s \in \mathcal{K}. \tag{21}$$

We assume that (x, s) is a strictly feasible solution of (19). We apply Newton's approach for (21). In fact, we want the new iterates $x + \Delta x$ and $s + \Delta s$ such that

$$\begin{gathered} s + \Delta s - M(x + \Delta x) - q = \nu r_q^0, \\ \psi(\tfrac{x \circ s}{\mu} + \tfrac{x \circ \Delta s + \Delta x \circ s + \Delta x \circ \Delta s}{\mu}) = \psi(e), \\ x + \Delta x, s + \Delta s \in \mathcal{K}. \end{gathered}$$

Neglecting the quadratic term $\Delta x \circ \Delta s$ and using Lemma 2.1, since $s - Mx - q = \nu r_q^0$, we obtain

$$\begin{gathered} \Delta s - M\Delta x = 0, \\ x \circ \Delta s + s \circ \Delta x = \mu\big(\psi'(\tfrac{x \circ s}{\mu})\big)^{-1} \circ \big(\psi(e) - \psi(\tfrac{x \circ s}{\mu})\big). \end{gathered} \tag{22}$$

3.2. **A new algorithm.** Initially, we have $\delta(x^0, s^0; \mu^0) = 0$. In what follows, we assume that at the start of each iteration, just before the μ-update, $\delta(x, s; \mu) \leq \tau$. So, this is certainly true at the start of the first iteration. Now suppose that the iterate (x, s) is strictly feasible of (19) for $\mu = \nu\mu^0$ and such that $\delta(x, s; \mu) \leq \tau$. We reduce μ to $\mu^+ = (1 - \theta)\mu$ and ν to $\nu^+ = (1 - \theta)\nu$, with $\theta \in (0, 1)$, and find displacements $\Delta^f x$ and $\Delta^f s$ such that

$$\begin{aligned} M\Delta^f x - \Delta^f s &= \theta\nu r_q^0, \\ P(w)^{\frac{1}{2}} s \circ P(w)^{-\frac{1}{2}} \Delta^f x + P(w)^{-\frac{1}{2}} x \circ P(w)^{\frac{1}{2}} \Delta^f s &= \\ 2\Big(\big(\mu P(w)^{-\frac{1}{2}} x \circ P(w)^{\frac{1}{2}} s\big)^{\frac{1}{2}} - P(w)^{-\frac{1}{2}} x \circ P(w)^{\frac{1}{2}} s\Big)&, \end{aligned} \tag{23}$$

where w is the NT-scaling point of x and s. It is easily seen that $x^f := x + \Delta^f x$ and $s^f := s + \Delta^f s$ satisfy the affine equation in (19), with $\nu = \nu^+$. Then, just by performing a centering step starting at (x^f, s^f) and targeting at μ^+-center of (19) with $\nu = \nu^+$, we obtain iterates (x^+, s^+) that are strictly feasible for (19) with $\nu = \nu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$. We define

$$d_x^f := \frac{P(w)^{-\frac{1}{2}} \Delta^f x}{\sqrt{\mu}},\ d_s^f := \frac{P(w)^{\frac{1}{2}} \Delta^f s}{\sqrt{\mu}}. \tag{24}$$

One can easily check that the system (23), which defines the search directions $\Delta^f x$ and $\Delta^f s$, can be written in terms of the scaled search directions d_x^f and

d_s^f as follows

$$\overline{M}d_x^f - d_s^f = \frac{\theta\nu}{\sqrt{\mu}}P(w^{\frac{1}{2}})r_q^0,\ d_x^f + d_s^f = p_v, \tag{25}$$

where $\overline{M} := P(w)^{\frac{1}{2}}MP(w)^{\frac{1}{2}}$ and $p_v := 2(e-v)$. Let $\tilde{p}_v := d_x^f - d_s^f$. Then, we have

$$d_x^f \circ d_s^f = \frac{p_v \circ p_v - \tilde{p}_v \circ \tilde{p}_v}{4}, \tag{26}$$

which implies that

$$\frac{\|\tilde{p}_v\|_F^2}{4} = \frac{\|p_v\|_F^2}{4} - \langle d_x^f, d_s^f\rangle. \tag{27}$$

A formal description of the algorithm is given as follows.

$Infeasible\ interior - point\ algorithm$

Input :
- accuracy parameter $\epsilon > 0$;
- barraier update parameter $\theta,\ 0 < \theta < 1$;
- threshold parameter $\tau > 0$;
- initialization parameters $\rho_{\rm p} > 0,\ \rho_{\rm d} > 0$.

begin
- $x := \rho_p e;\ s := \rho_d e;\ \mu^0 := \rho_p\rho_d$;
- **while** $\max\left(\mu N, \nu\|r_q^0\|_F\right) > \epsilon$
  - $(x, s) := (x, s) + (\Delta^f x, \Delta^f s)$;
  - μ and $\nu -$ update :
    - $\mu := (1-\theta)\mu;\ \nu := (1-\theta)\nu$;
    - $(x, s) := (x, s) + (\Delta x, \Delta s)$;
- **end while**

end.

3.3. **Analysis of the algorithm.** Let $x^f = x + \Delta^f x$ and $s^f = s + \Delta^f s$ be the iterates obtained after the feasibility step. Then, by using (24), we have

$$x^f = \sqrt{\mu}P(w)^{\frac{1}{2}}(v + d_x^f),\ s^f = \sqrt{\mu}P(w)^{-\frac{1}{2}}(v + d_s^f).$$

Since $P(w)^{\frac{1}{2}}$ and its inverse $P(w)^{-\frac{1}{2}}$ are automorphisms of $\mathcal{K}$, the iterates x^f and s^f belong to int$\mathcal{K}$ if and only if $v + d_x^f$ and $v + d_s^f$ belong to int$\mathcal{K}$, respectively. Moreover, $p_v = 2(e - v)$ implies that

$$v \circ v + v \circ p_v = e - \frac{1}{4}p_v \circ p_v. \tag{28}$$

In what follows, we use the notation $\bar{\omega} := \frac{1}{2}\sqrt{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}$. In the next lemma we give a condition in terms of $\delta(v)$ and $\bar{\omega}$, which guarantees the feasibility of x^f and s^f.

Lemma 3.2. *The iterate (x^f, s^f) is strictly feasible if $\delta(v)^2 + 2\bar{\omega}^2 < 1$.*

Proof. We define $v_x(\alpha) := v + \alpha d_x^f$ and $v_s(\alpha) := v + \alpha d_s^f$, for $0 \le \alpha \le 1$. We thus have

$$\begin{aligned} v_x(\alpha) \circ v_s(\alpha) &= v^2 + \alpha v \circ (d_x^f + d_s^f) + \alpha^2 d_x^f \circ d_s^f \\ &= (1-\alpha)v^2 + \alpha(v^2 + v \circ p_v) + \alpha^2\Big(\frac{p_v \circ p_v - \tilde{p}_v \circ \tilde{p}_v}{4}\Big) \\ &= (1-\alpha)v^2 + \alpha\Big(e - (1-\alpha)\frac{p_v \circ p_v}{4} - \alpha\frac{\tilde{p}_v \circ \tilde{p}_v}{4}\Big). \end{aligned}$$

It follows that $v_x(\alpha) \circ v_s(\alpha) \in \mathrm{int}\mathcal{K}$ holds if

$$\Big\|(1-\alpha)\frac{p_v \circ p_v}{4} + \alpha\frac{\tilde{p}_v \circ \tilde{p}_v}{4}\Big\|_F < 1.$$

Using the triangle inequality and (27) we obtain

$$\begin{aligned} \Big\|(1-\alpha)\frac{p_v \circ p_v}{4} + \alpha\frac{\tilde{p}_v \circ \tilde{p}_v}{4}\Big\|_F &\le (1-\alpha)\Big\|\frac{p_v \circ p_v}{4}\Big\|_F + \alpha\Big\|\frac{\tilde{p}_v \circ \tilde{p}_v}{4}\Big\|_F \\ &\le (1-\alpha)\frac{\|p_v\|_F^2}{4} + \alpha\frac{\|\tilde{p}_v\|_F^2}{4} = \delta(v)^2 - \alpha\langle d_x^f, d_s^f\rangle \le \delta(v)^2 + 2\bar{\omega}^2, \end{aligned}$$

where the last inequality follows due to $0 < \alpha \le 1$ and the following inequality

$$-\langle d_x^f, d_s^f\rangle \le |\langle d_x^f, d_s^f\rangle| \le \|d_x^f\|_F\|d_s^f\|_F \le \frac{1}{2}\big(\|d_x^f\|_F^2 + \|d_s^f\|_F^2\big) = 2\bar{\omega}^2.$$

Therefore, the assumption $\delta(v)^2 + 2\bar{\omega}^2 < 1$ implies that $v_x(\alpha) \circ v_s(\alpha) \in \mathrm{int}\mathcal{K}$ for $0 \le \alpha \le 1$. Hence, since $x, s \in \mathrm{int}\mathcal{K}$, Lemma 2.2 implies that $v_x(1) = v + d_x^f \in \mathrm{int}\mathcal{K}$ and $v_s(1) = v + d_s^f \in \mathrm{int}\mathcal{K}$. This completes the proof. □

Let

$$v^f := \frac{P((w^f)^{-\frac{1}{2}})x^f}{\sqrt{\mu^+}}\Big[= \frac{P((w^f)^{\frac{1}{2}})s^f}{\sqrt{\mu^+}}\Big],$$

where w^f is the NT-scaling point of x^f and s^f. In the sequel, we denote $\delta(x^f, s^f; \mu^+)$ shortly by $\delta(v^f)$.

Lemma 3.3. *If $\delta(v)^2 + 2\bar{\omega}^2 < 1$. Then*

$$\delta(v^f) \le \frac{\delta(v)^2 + 2\bar{\omega}^2 + \theta\sqrt{2N}}{1 - \theta + \sqrt{(1-\theta)(1-\delta(v)^2 - 2\bar{\omega}^2)}}.$$

Proof. The proof of the lemma is similar to the proof of Lemma 12 in [10]. □

Lemma 3.4. *Let* $\delta(v) \leq \tau < 1$. *Then* $1-\tau \leq \lambda_i(v) \leq 1+\tau, i = 1, \ldots, 2N$.

Proof. From $\delta(v) = \|e-v\|_F \leq \tau$, we obtain

$$(\lambda_i(v)-1)^2 \leq \|v-e\|_F^2 \leq \tau^2, i = 1, \ldots, 2N.$$

This implies the desired result. □

Lemma 3.5. *If* SOCLCP *is the Cartesian* $P_*(\kappa)$*-property, then for any* $a, \tilde{b}$ *the linear system*

$$-\overline{M} d_x^f + d_s^f = \tilde{b}, \; d_x^f + d_s^f = a, \tag{29}$$

has a unique solution (d_x^f, d_s^f) *and the following inequality is satisfied:*

$$\left\|(d_x^f, d_s^f)\right\|_F \leq \sqrt{1+2\kappa}\|a\|_F + \left(1+\sqrt{2+4\kappa}\right)\eta(\tilde{b}),$$

where

$$\eta(\tilde{b})^2 = \min\left\{\left\|(\tilde{d}_x^f, \tilde{d}_s^f)\right\|_F^2 : -\overline{M}\tilde{d}_x^f + \tilde{d}_s^f = \tilde{b}\right\} = \tilde{b}^T(\overline{M}P(w)^{-1}\overline{M}^T + P(w))^{-1}\tilde{b}.$$

Proof. The proof of the lemma is similar to the proof of Lemma 3.3 in [13], and is therefore omitted. □

Comparing system (29) with the system (25) and considering $a = p_v$ and $\tilde{b} = -\frac{\theta\nu}{\sqrt{\mu}}P(w^{\frac{1}{2}})r_q^0$ in the system (29), we have

$$\begin{aligned}
\|d_x^f\|_F^2 + \|d_s^f\|_F^2 &\leq \left(\sqrt{1+2\kappa}\|p_v\|_F + \left(1+\sqrt{2+4\kappa}\right)\frac{\theta\nu}{\sqrt{\mu}}\eta\big(-P(w^{\frac{1}{2}})r_q^0\big)\right)^2 \\
&\leq \left(2\sqrt{1+2\kappa}\tau + \left(1+\sqrt{2+4\kappa}\right)\frac{\theta\nu}{\sqrt{\mu}}\eta(-P(w^{\frac{1}{2}})r_q^0)\right)^2. \qquad (30)
\end{aligned}$$

Let (x^*, s^*) be the optimal solution of the Cartesian $P_*(\kappa)$-SOCLCP that satisfies (17) and the algorithm starts with $(x^0, s^0) = (\rho_p e, \rho_d e)$. Then,

$$x^* - x^0 \preceq_{\mathcal{K}} \rho_p e, \; s^* - s^0 \preceq_{\mathcal{K}} \rho_d e, \tag{31}$$

$$\begin{aligned}
-P(w^{\frac{1}{2}})r_q^0 &= -P(w^{\frac{1}{2}})(s^0 - Mx^0 - q) \\
&= -P(w^{\frac{1}{2}})MP(w^{\frac{1}{2}})P(w^{-\frac{1}{2}})(x^* - x^0) + P(w^{\frac{1}{2}})(s^* - s^0) \\
&= -\overline{M}P(w^{-\frac{1}{2}})(x^* - x^0) + P(w^{\frac{1}{2}})(s^* - s^0). \qquad (32)
\end{aligned}$$

Now, by using the definition of $\eta(-P(w^{\frac{1}{2}})r_q^0)$, (31) and (32), we have

$$\begin{aligned}
\eta(-P(w^{\frac{1}{2}})r_q^0)^2 &\leq \left\|P(w^{-\frac{1}{2}})(x^* - x^0)\right\|_F^2 + \left\|P(w^{\frac{1}{2}})(s^* - s^0)\right\|_F^2 \\
&\leq \rho_p^2 \mathrm{tr}(w^{-2}) + \rho_d^2 \mathrm{tr}(w^2) \leq \rho_p^2 \frac{\mathrm{tr}(s^2)}{\mu\lambda_{\min}(v)^2} + \rho_d^2 \frac{\mathrm{tr}(x^2)}{\mu\lambda_{\min}(v)^2} \\
&\leq \rho_p^2 \frac{\mathrm{tr}(s)^2}{\mu(1-\tau)^2} + \rho_d^2 \frac{\mathrm{tr}(x)^2}{\mu(1-\tau)^2}. \qquad (33)
\end{aligned}$$

The third inequality follows by Lemma 4.5 in [8] and the last inequality follows by (12) and $\mathrm{tr}(z^2) \leq \mathrm{tr}(z)^2$ for each $z \in \mathcal{K}$.

Lemma 3.6. *Let* (x, s) *be feasible for the perturbed problem* (19) *and let* $(x^0, s^0) = (\rho_p e, \rho_d e)$ *and* (x^*, s^*) *be as defined in (17). Then,*

$$\mathrm{tr}(x) \leq 2N(1 + 4\kappa)\rho_p(2 + (1 + \tau)^2), \ \mathrm{tr}(s) \leq 2N(1 + 4\kappa)\rho_d(2 + (1 + \tau)^2).$$

Proof. It is easily seen that

$$\nu s^0 + (1 - \nu)s^* - s = M(\nu x^0 + (1 - \nu)x^* - x).$$

From the Cartesian $P_*(\kappa)$ property of M, we get

$$\begin{aligned}
&\langle \nu x^0 + (1 - \nu)x^* - x, \nu s^0 + (1 - \nu)s^* - s\rangle \\
&\qquad \geq -4\kappa \sum_{j \in I_+} \Big(\langle \nu x_j^0 + (1 - \nu)x_j^* - x_j, \nu s_j^0 + (1 - \nu)s_j^* - s_j\rangle \Big) \\
&\qquad \geq -4\kappa \sum_{j \in I_+} \Big(\nu^2 \langle x_j^0, s_j^0\rangle + \nu(1 - \nu)\big(\langle x_j^0, s_j^*\rangle + \langle x_j^*, s_j^0\rangle\big) + \langle x_j, s_j\rangle \Big),
\end{aligned}$$

$$\geq -4\kappa \sum_{j=1}^{N} \Big(\nu^2 \langle x_j^0, s_j^0\rangle + \nu(1 - \nu)\big(\langle x_j^0, s_j^*\rangle + \langle x_j^*, s_j^0\rangle\big) + \langle x_j, s_j\rangle \Big), \tag{34}$$

where the second inequality follows by $\langle x^0, s\rangle + \langle x, s^0\rangle \geq 0$, $\langle x^*, s\rangle + \langle x, s^*\rangle \geq 0$ and $\langle x^*, s^*\rangle = 0$. By rearranging the above inequality and using $x^0 = \rho_p e, s^0 = \rho_d e, \|x^*\|_\infty \leq \rho_p, \|s^*\|_\infty \leq \rho_d$ and $\langle x, s\rangle = \mu\langle v, v\rangle \leq 2N\mu(1 + \tau)^2$, we obtain

$$\begin{aligned}
\langle x^0, s\rangle + \langle x, s^0\rangle \leq& (1 + 4\kappa)\Big(\nu\langle x^0, s^0\rangle + (1 - \nu)\big(\langle x^0, s^*\rangle + \langle x^*, s^0\rangle\big) + \frac{1}{\nu}\langle x, s\rangle\Big) \\
\leq& (1 + 4\kappa)\Big(2N\nu\rho_p\rho_d + 4N(1 - \nu)\rho_p\rho_d + 2N\rho_p\rho_d(1 + \tau)^2\Big) \\
\leq& (1 + 4\kappa)2N\rho_p\rho_d\big(2 + (1 + \tau)^2\big).
\end{aligned}$$

Therefore, $\langle x, s^0\rangle \leq (1 + 4\kappa)2N\rho_p\rho_d\big(2 + (1 + \tau)^2\big)$ which implies the result. □

Using Lemma 3.6, (33), (30) and $\mu = \nu\rho_p\rho_d$, we obtain

$$\|d_x^f\|_F^2 + \|d_s^f\|_F^2 \leq \Big(2\sqrt{1 + 2\kappa}\tau + 2\sqrt{2}N\theta(1 + 4\kappa)\big(1 + \sqrt{2 + 4\kappa}\big)\frac{2 + (1 + \tau)^2}{1 - \tau}\Big)^2.$$

Therefore, by the definition of $\bar{\omega}$, we get

$$\bar{\omega} \leq \sqrt{1 + 2\kappa}\tau + \sqrt{2}N\theta(1 + 4\kappa)\big(1 + \sqrt{2 + 4\kappa}\big)\frac{2 + (1 + \tau)^2}{1 - \tau}. \tag{35}$$

In this stage, we choose $\tau = \frac{1}{16(1+4\kappa)}$ and $\theta = \frac{1}{27N(1+4\kappa)^2}$. From (35) it follows that $\bar{\omega} < \frac{1}{2\sqrt{1+4\kappa}}$. Moreover, $\delta(v)^2 + 2\bar{\omega}^2 < \frac{1}{256(1+4\kappa)^2} + \frac{1}{2(1+4\kappa)} < 1$, which implies that the iterate (x^f, s^f) is a strictly feasible solution of (19) with $\nu = \nu^+$. The next lemma gives an upper bound for $\delta(v^f)$.

Lemma 3.7. *Let $\delta(v) \le \tau$. Then, $\delta(v^f) < \frac{0.3363}{1+4\kappa}$.*

Proof. From Lemma 3.3 we have

$$\begin{aligned}\delta(v^f) &\le \frac{\delta(v)^2 + 2\bar{\omega}^2 + \theta\sqrt{2N}}{1-\theta+\sqrt{(1-\theta)(1-\delta(v)^2-2\bar{\omega}^2)}}\\ &\le \frac{\tau^2 + 2\bar{\omega}^2 + \theta\sqrt{2N}}{1-\theta+\sqrt{(1-\theta)(1-\tau^2-2\bar{\omega}^2)}}.\end{aligned}$$

Now, using $\tau = \frac{1}{16(1+4\kappa)}, \theta = \frac{1}{27N(1+4\kappa)^2}$ and $\bar{\omega} < \frac{1}{2\sqrt{1+4\kappa}}$, we get

$$\begin{aligned}\delta(v^f) &< \frac{(\frac{1}{16(1+4\kappa)})^2 + 2(\frac{1}{2\sqrt{1+4\kappa}})^2 + \frac{\sqrt{2N}}{27N(1+4\kappa)^2}}{1-\frac{1}{27N(1+4\kappa)^2}+\sqrt{(1-\frac{1}{27N(1+4\kappa)^2})(1-(\frac{1}{16(1+4\kappa)})^2-2(\frac{1}{2\sqrt{1+4\kappa}})^2)}}\\ &\le \frac{\frac{1}{1+4\kappa}\big(\frac{1}{16^2}+\frac{1}{2}+\frac{\sqrt{2}}{27}\big)}{\frac{26}{27}+\sqrt{\frac{26}{27}\big(1-\frac{1}{16^2}-\frac{1}{2}\big)}} \le \frac{0.3363}{1+4\kappa}.\end{aligned}$$

This implies the desired result. □

Lemma 3.8. *Let (x^+, s^+) be the iterates obtained by a main iteration of the algorithm and $\delta(v) \le \tau$. Then $\delta(v^+) := \delta(x^+, s^+; \mu^+) < \frac{1}{16(1+4\kappa)}$.*

Proof. Since the iterate (x^+, s^+) is obtained by a main iteration of the algorithm, thus $x^+ = x^f + \Delta x,\ s^+ = s^f + \Delta s$. Using Lemma 3.7, we have

$$\delta(v^f) < \frac{0.3363}{1+4\kappa} < \frac{1}{\sqrt{1+4\kappa}},$$

which applying Lemma 2.5 for (19) with $\nu = \nu^+$ implies that x^+ and s^+ are strictly feasible. Now, we use Lemma 2.8 for (19) with $\nu = \nu^+$ and we obtain

$$\delta(v^+) \le \frac{(1+4\kappa)\delta(v^f)^2}{1+\sqrt{1-(1+4\kappa)\delta(v^f)^2}} < \frac{1}{16(1+4\kappa)}.$$

This completes the proof. □

In each main iteration, both the duality gap and the norm of the residual are reduced by the factor $1-\theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta}\log\frac{\max\{(x^0)^T s^0, \|r_q^0\|_F\}}{\epsilon}.$$

Since every main iteration consists of two inner iterations, we may state the main result of the paper.

Theorem 1. *If* (1) *has an optimal solution* (x^*, s^*) *such that* $\|x^*\|_\infty \leq \rho_p$ *and* $\|s^*\|_\infty \leq \rho_d$*, for some* $\rho_p, \rho_d > 0$*, then after at most*

$$54N(1+4\kappa)^2 \log \frac{\max\{(x^0)^T s^0, \|r_q^0\|_F\}}{\epsilon}$$

iterations, the algorithm finds an ϵ*-optimal solution of the Cartesian* $P_*(\kappa)$*-SOCLCP.*

4. Conclusions

We proposed and analyzed a new full Nesterov-Todd step infeasible interior-point method for the Cartesian $P_*(\kappa)$-SOCLCP based on the technique introduced in [5]. We have shown that in each iteration the new algorithm needs a feasibility step and one centering step in order to prove that the algorithm is well defined. We derived the complexity bound for the algorithm which coincides with the currently best-known iteration bound for IIPMs.

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