

DEGREE SETS OF TOURNAMENTS

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ABSTRACT. The *score set* of a tournament is defined as the set of its different outdegrees. In 1978 Reid [20] published the conjecture that for any set of nonnegative integers D there exists a tournament T whose degree set is D . Reid proved the conjecture for tournaments containing $n = 1, 2$ and 3 vertices. In 1986 Hager [7] published a constructive proof of the conjecture for $n = 4$ and 5 vertices. Yao [27] in 1989 presented an arithmetical proof of the conjecture, but general polynomial construction algorithm is not known. In [11] we described polynomial time algorithms which reconstruct the score sets containing only elements less than 7 .

In this paper we present and analyze earlier proposed algorithms BALANCING and SHORTENING, further new algorithms SHIFTING and HOLE which together reconstruct the score sets containing elements less than 9 and so give a constructive partial proof of the Reid conjecture.

1. BASIC DEFINITIONS

We will use the following definitions [6].

A **graph** $G(V, E)$ consists of two finite sets V and E , where the elements of V are called **vertices**, the elements of E are called **edges** and each edge has a set of one or two vertices associated to it, which are called its **endpoints** (**head and tail**). An edge is said to **join** its endpoints. A **simple graph** is a graph that has no self-loops and multi-edges.

A **directed edge** is said to be **directed** from its **tail** and **directed** to its **head**. (The tail and head of a directed self-loop are the same vertex.)

A **directed graph** (**shortly: digraph**) is a graph whose each edge is directed. If in a directed graph $(u, v) \in E$, then we say that u **dominates** v . An **oriented graph** is a digraph obtained by choosing an orientation (direction) for each edge of a simple graph. A **tournament** is a complete oriented graph. That is, it has no self-loops, and between every pair of vertices,

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there is exactly one edge. Beside the terms of graph theory we will use the popular terms player, score sequence, score set, point, win, loss etc.

A directed graph (so a tournament too) $F = (E, V)$ is **transitive**, if $(u, v) \in E$ and $(v, w) \in E$ imply $(u, w) \in E$.

The **order of a tournament** T is the number of vertices in T . A tournament of order n will be called an n -tournament.

An (a, b, n) -**tournament** is a loopless directed graph, in which every pair of distinct vertices is connected with at least a and at most $b \geq a$ edges.

The **score** (or **out-degree**) of a vertex v in a tournament T is the number of vertices that v dominates. It is denoted by $d_T^+(v)$ (shortly: $d(v)$).

The **degree sequence** (score sequence) of an n -tournament T is the ordered n -tuple s_1, s_2, \dots, s_n , where s_i is the score of the vertex v_i , $1 \leq i \leq n$, and

$$(1) \quad s_1 \leq s_2 \leq \dots \leq s_n.$$

The **score set** of an n -tournament T is the ordered m -tuple $D = (d_1, d_2, \dots, d_m)$ of the different scores of T , where

$$(2) \quad d_1 < d_2 < \dots < d_m.$$

2. INTRODUCTION

Theorem Landau [12] allows to test potential score sequences in linear time.

Theorem 1. (Landau [12]) *A nonincreasing sequence of nonnegative integers $S = s_1, s_2, \dots, s_n$ is a score sequence of an n -tournament if and only if*

$$(3) \quad \sum_{i=1}^k s_i \geq \frac{k(k-1)}{2}, \quad 1 \leq k \leq n,$$

with equality when $k = n$.

Proof. See [12, 21]. □

Beineke and Eggleton [21, p. 180] noted in the 1970's that not all of the Landau inequalities need to be checked when testing a sequence S for realizability as a score sequence of some tournament. One only need to check

$$(4) \quad \sum_{i=1}^k s_i \geq \binom{k}{2}$$

for those values of k for which $s_k < s_{k+1}$. In 2003 Tripathi and Vijay proved this assertion [24].

To reconstruct a prescribed score set is much harder problem, then computing the score set belonging to the score sequence of a given tournament.

Therefore surprising is the following conjecture published by Reid in 1978 [20]: if $m \geq 1$, $D = \{d_1, d_2, \dots, d_m\}$ is a set of nonnegative integers, then there exists a tournament whose score set is D .

In his paper Reid described the proof of his conjecture for score sets containing 1, 2, and 3 elements, further for score sets representing an arithmetical or geometrical series. In 1986 Hager [7] proved the conjecture for $m = 4$ and $m = 5$.

In 2006 Pirzada and Naikoo [17] gave a constructive proof of a new special case of Theorem 5.

Theorem 2. (Pirzada and Naikoo [17]) *If s_1, s_2, \dots, s_m are nonnegative integers with $s_1 < s_2 < \dots < s_m$, then there exists such $n \geq m$ for which there exists an n -tournament T with score set*

$$(5) \quad D = \left\{ d_1 = s_1, d_2 = \sum_{i=1}^2 s_i, \dots, d_m = \sum_{i=1}^m s_i \right\}.$$

Proof. See [17]. □

In [18] Pirzada and Naikoo characterized the score sets of k -partite hypertournaments, and in 2008 the score sets of oriented graphs.

Theorem 3. (Pirzada, Naikoo [19]) *Let a, ad, ad^2, \dots, ad^n , where a, d and n are positive integers with $d > 1$. Then there exists an oriented graph with score set A except for $a = 1, d = 2$ and $n > 0$ and for $a = 1, d = 3$ and $n > 0$.*

Proof. See [19]. □

The following theorem contains a sufficient condition of the existence of oriented graphs with special prescribed score set.

Theorem 4. (Pirzada, Naikoo [19]) *If a_1, a_2, \dots, a_n is an increasing sequence of nonnegative integers, then there exists an oriented graph with $a_n + 1$ vertices and score set D , where*

$$(6) \quad a_i = \begin{cases} a_i, & \text{if } i = 1, \\ a_{i-1} + a_i + 1, & \text{if } i > 1 \end{cases}$$

Proof. See [19]. □

In 1989 Yao proved the conjecture of Reid.

Theorem 5. (Yao [27]) *If $m \geq 1$, $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$ is a set of nonnegative integers, then there exists a score sequence $S = s_1, s_2, \dots, s_n$ such, that the score set of the tournament belonging to S is D .*

Proof. See [27]. □

The proof of Yao uses arithmetical tools and only proves the existence of the corresponding tournaments, but it does not give a construction.

In 1983 Wayland [25] proposed a sufficient condition for a set D of non-negative integers to be the score set of a bipartite tournament. This result was improved to a sufficient and necessary condition in 1983 by Petrović [14].

In [10] we proved that the extension of Yao's theorem is not true for k -tournaments (where every pair of vertices is connected with $k \geq 2$ edges).

Recently we proposed algorithms BALANCING and SHORTENING [11] and proved Yao's theorem for score sets containing only elements less than 7. In this paper we describe new algorithms SHORTENING and HOLE and prove Theorem 5 for sets containing elements less than 9. Our proofs are constructive and the reconstruction algorithms require only polynomial time.

Now we present three lemmas allowing a useful extension of Theorem 5.

Lemma 1. *If $d_1 \geq 1$, then the score set $D = \{d_1\}$ is realizable by the unique score sequence $S = d_1^{<2d_1+1>}$.*

Proof. If $|S| = n$ and S generates D then the sum of the elements of S equals to nd_1 and also to $n(n-1)/2$ implying $n = 2d_1 + 1$. Such tournament is realizable for example so, that any player P_i gathers one points against players $P_{i+1}, \dots, P_{i+(n-1)/2}$ and zero against the remaining players (the indices are mod n taken). \square

In this lemma and later $a^{}$ means a multiset, in which a is repeated b times.

Lemma 2. *If the score sequence $S = s_1, s_2, \dots, s_n$ corresponds to the score set $D = \{d_1, d_2, \dots, d_m\}$, then $n \geq d_m + 1$.*

Proof. If the score of a vertex v is d_m , then v dominates d_m different vertices. \square

Lemma 3. *If $m \geq 2$ and the score sequence $S = s_1, s_2, \dots, s_n$ corresponds to the score set $D = \{d_1, d_2, \dots, d_m\}$, then*

$$(7) \quad 2d_1 + 2 \leq n \leq 2d_m,$$

and both bounds are sharp.

Proof. Every element of D has to appear in S . Therefore the arithmetical medium of the scores is greater, than d_1 , and smaller, than d_m . From the other side n -tournaments contain $B_n = \binom{n}{2}$ edges, so the arithmetical medium of their scores is $B_n/n = (n-1)/2$, therefore

$$(8) \quad d_1 < \frac{n-1}{2} < d_m,$$

implying (7).

For example if $k \geq 0$ and $D = \{k, k+1\}$, then according to (8) $n = 2k + 2$ imply the sharpness of both the bounds. \square

The next extension of Theorem 5 is based on Lemmas 1, 2, 3.

Theorem 6. (Iványi, Lucz, Gombos, Matuszka [11]) *If $m \geq 1$ and $D = \{d_1, d_2, \dots, d_m\}$ is an increasingly ordered set of nonnegative integers, then*

- *there exist a tournament T , whose score sequence is S and score set is D ;*
- *if $m = 1$, then $S = s_1^{<2d_1+1>}$;*
- *if $m \geq 2$, then*

$$(9) \quad \max(d_m + 1, 2d_1 + 2) \leq n \leq 2d_m;$$

- *the bounds in (9) are sharp.*

Proof. The assertion follows from the above lemmas (see [11]). \square

Taking into account the remark of Beineke and Eggleton [21, page 180] we can formulate Reid's conjecture as an arithmetical statement without the terms of the graph theory. Let $D = \{d_1, d_2, \dots, d_m\}$ be an increasingly ordered set of nonnegative integers. According to the conjecture there exist positive integer exponents x_1, x_2, \dots, x_m such that

$$(10) \quad S = d_1^{<x_1>}, d_2^{<x_2>}, \dots, d_m^{<x_m>}$$

is the score sequence of some $(\sum_{i=1}^m x_i)$ -tournament. Using Landau's theorem it can be easily seen that Reid's conjecture is equivalent to the following statement [16, 27].

For every $(0, d_m, m)$ -regular set $D = \{d_1, \dots, d_m\}$ there exist positive integers x_1, \dots, x_m , such that

$$(11) \quad \sum_{i=1}^k x_i d_i \geq \binom{\sum_{i=1}^k x_i}{2}, \quad \text{for } k = 1, \dots, m-1,$$

and

$$(12) \quad \sum_{i=1}^m x_i d_i = \binom{\sum_{i=1}^m x_i}{2}.$$

Commenting Yao's proof Qiao Li wrote in 1989 [13]: *Yao's proof is the first proof of the conjecture, but I do not think it is the last one. I hope a shorter and simpler new proof will be coming in the near future.*

However, the constructive proof has not been discovered yet.

Our algorithms investigate only the zerofree score sets, The base of this approach is the following lemma.

The base of considering only the zerofree sets is the following assertion.

Lemma 4. *Let $m \geq 2$. A sequence $S = s_1^{<e_1>}, s_2^{<e_2>}, \dots, s_n^{<e_m>}$ is the score sequence corresponding to the score set $D = \{0, d_2, d_3, \dots, d_m\}$ if and only*

if the sequence $S' = (s_2 - 1)^{\langle e_2 \rangle}, (s_3 - 1)^{\langle e_3 \rangle}, \dots, (s_n - 1)^{\langle e_n \rangle}$ is the score sequence corresponding to $D' = \{d_2 - 1, d_3 - 1, \dots, d_m - 1\}$.

Proof. If S is the score sequence corresponding to D then $s_1 = 0$ and $e_1 = 1$ that is all other players won against the player having the score $s_1 = 0$, so S' corresponds to D' .

If S' does not correspond to D' , then we add a new score $d_1 = 0$ to D' , increase the multiplicity of the other scores by 1 and get D which does not correspond to S . \square

3. RECONSTRUCTION OF SCORE SETS OF TOURNAMENTS

Earlier we proposed [11] polynomial approximate algorithms BALANCING and SHORTENING, further exponential exact algorithms SEQUENCING and DIOPHANTINE to reconstruct score sets. Now we add approximate algorithms SHIFTENING and HOLE. The polynomial algorithms are based on Theorem 6.

Since there are quick (quadratic) algorithms constructing n -tournaments corresponding to a given score sequence, our algorithms construct only a suitable score sequence.

If the score sequence of a tournament is S and its score set is D , then we say, that S generates D , or D corresponds to S . If D is given, then we call the corresponding score sequence **good**.

3.1. Concept of the BALANCING algorithm. The main idea behind BALANCING algorithm [11] is that each element of D can be classified as a **winner**, **loser** or **balanced** score. We say that $d_i \in D$ is a winner if it is greater than half of the number of players. Conversely, it is a loser if it is less than $n/2$. In case of equality d_i is a balanced element. Let *plus* be the difference between a winner's score and its number of lost matches, $plus_i = d_i - ((n - 1) - d_i)$, and *minus* _{i} the difference between its number of won matches and its scores, that is $minus_i = ((n - 1) - d_i) - d_i$. It is obvious that the sum of *plus* _{i} 's and the sum of *minus* _{i} 's in a tournament must be equal. Therefore if we choose the number of losers and the number of winners such that this condition holds, then we can create a potential score sequence. Notice that a score can be classified only if the number of players is fixed. For this purpose we use Theorem 6, where based on the score set D we can find a finite set of possible n 's. Moreover a potential score sequence must correspond to the Landau theorem.

3.2. Concept of the SHORTENING algorithm. The BALANCING algorithm successfully finds a good sequence while the greatest element of the score set is not greater, than 5. If $d_m > 5$, then it seems to be a good idea to remove some scores from the set and search a sequence corresponding to the shortened set.

We are listing now the three different shortening procedures which are used in the SHORTENING algorithm:

1) Consider a score set D , where $d_m = n - 1$ and the equation $d_{m-k} = d_{m-k+1} - 1$ is also true for every $k \geq 0$ and $k < m$. In that case there have to be a player with d_m score who defeats all the others, so it can be removed without changing the other's scores. If $k > 0$ then, after removing d_m , the largest element of D is also defeats everyone (except the player with the removed score). So one can remove the last $k + 1$ elements in such a case and search a good sequence for the shortened score set. After a valid sequence is found the discarded elements can be append to the shortened set as those defeat everyone.

2) The previous procedure can be done also in the other way. If the first element of D is zero then there have to be a player who is defeated by everyone. This player can be removed while the other's scores is decreased by one. Moreover the first l score can be discarded if $d_1 = 0$ and $d_j = d_{j-1} + 1$ for every $j \leq l$. As before if one can found a good score sequence for the shortened set, the discarded elements can be inserted based on Lemma 4.

3) If none of the above methods results a valid sequence we can try to aggressively remove elements from the score set. If we exclude the last element of D and can find a valid sequence for the shortened set, then there are some hope that we can construct a sequence corresponding to the original set by appending the discarded element one or more times to the short sequence. If the sequence do not conform to the Landau theorem after $n - m$ appended elements we stop and conclude that the algorithm was not able to find a valid sequence.

3.3. Concept of the SHIFTENING algorithm. In some situations one can find a good sequence for a shortened score set by shifting the scores. Consider two score sets, the original $D = \{d_1, d_2, \dots, d_m\}$ and the shortened $D' = \{d_1, d_2, \dots, d_{m-1}\}$. If we find a sequence $S = \{s_1, s_2, \dots, s_{n'}\}$ corresponding to the shortened set, then we have three possibilities to construct a sequence that generates D .

1) $d_m = n$. In this simple case we add one more player that defeats everyone. As the number of players in S' equal to the required score d_m we get a valid sequence $S = S' \cup \{d_m\}$ generating D .

2) $d_m > n$. Let $diff = d_m - n$ then we can add $N_{new} = 2diff + 1$ new players with score d_m . These defeat all the original players by which they gain n points. The remaining points are obtained by the new players through defeating each other $diff$ times. So the sequence $S = S' \cup_{i=1}^{N_{new}} \{d_m\}$ is valid.

3) $d_m < n$. In this case there is no exact way to find S but we can try to split the new player's additional points. Let $diff = n - d_m$ be the additional

points and $X' = \{x_1, x_2, \dots, x_{m-1}\}$ be the exponents of scores in the set D' . If we can find an integer i such that $d_{i+1} - d_i = i$ and $x_i > 1$ then we can give a point to one of the players with s_i and take away one from the new player. With our notation this can be written as $x_i = x_i - 1$ and $x_{i+1} = x_{i+1} + 1$ while $s_{n+1} = s_{n+1} - 1$. Also we can decrease *diff* by one. This method can be repeated while *diff* > 0 and if *diff* $= 0$, then we are done and the exponents $X = \{x_1, x_2, \dots, x_{m-1}, 1\}$ generate a corresponding score sequence.

3.4. Concept of the HOLE algorithm. While the maximum score in D is not greater, than 7, the three algorithms above find a corresponding sequence every time. Moreover if we set this number to 8, then the algorithms return a sequence for all sets except two. However, these two sets have the common property of holes. In a score set $D = \{d_1, d_2, \dots, d_m\}$ there is a *single hole* at $1 \leq j \leq d_m - 1$, if j is missing from D , but $j - 1$ and $j + 1$ are contained by D . If $d_1 = 1$, then the absence of 0 is also a hole. For the mentioned two sets it is true that there is no j such that $d_{j+1} - d_j > 2$ and $d_1 = 1$. So these sets only differ in holes from the score sequence of the transitive $(d_m + 1)$ -tournament $S = \{0, 1, \dots, m\}$ set. HOLE compares D with S and handles the hole at 0 by changing the result of the match between the players having 0 and $d_m + 1$ points and increasing the exponents of the scores 1 and d_m . The holes at $1 \leq j \leq d_m - 1$ are handled by changing the result between the players having scores j and $j + 1$ and increasing the exponents e_{j-1} and e_{j+1} by 1.

The running time of HOLE is $\Theta(m)$.

4. SCORE SEQUENCES PRODUCED BY BALANCING, SHORTENING, SHIFTENING, AND HOLE

In this subsection we present the score sequences produced by the approximate algorithms BALANCING, SHORTENING, SHIFTENING, and HOLE, if their input data are the zerofree score sets with $d_m \leq 8$.

BALANCED reconstructs each score set characterized by $d_m \leq 5$, and 30 zerofree score sets characterized by $d_m = 6$ (due to Lemma 4 it is sufficient to investigate only the zerofree sets). The exceptional score sets are $\{1, 3, 6\}$ and $\{1, 2, 3, 5, 6\}$, which are reconstructed by SHORTENING. So these algorithms together reconstruct each score set characterized by $d_m \leq 6$, but according to Tables 1 and 2 they can not reconstruct the score sets $\{1, 2, 3, 5, 7\}$ and $\{1, 2, 3, 4, 6, 7\}$. According to Table 2 SHIFTENING reconstructs the set $\{1, 2, 3, 5, 7\}$, and according to [4, 5] HOLE finds the sequence of exponents 2,1,1,2,1,2, which determine a score sequence corresponding to $\{1, 2, 3, 4, 6, 7\}$. We received that these three approximate algorithms together reconstruct each score set satisfying the condition $d_m \leq 7$.

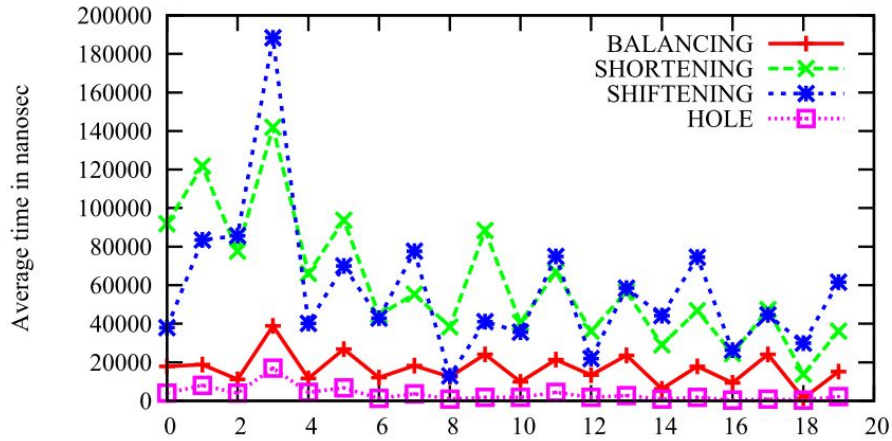


FIGURE 1. The running time of the approximate algorithms.

According to [4, 5] BALANCED, SHORTENING and SHIFTENING together reconstruct the majority of the zerofree score sequences with $d_m \leq 8$. Exceptions are only the sets $1, 2, 3, 5, 7, 8$ and $1, 2, 3, 4, 6, 7, 8$.

Both of these sets contain only single holes, therefore HOLE easily reconstructs them. E.g. $1, 2, 3, 5, 7, 8$ contains single holes at $j = 3$ and $j = 5$. Using HOLE we get the sequence of exponents $2, 1, 2, 1, 2, 2$. $D = \{1, 2, 3, 4, 6, 7, 8\}$ contains a single hole at $j = 5$. Using HOLE we get the sequence of exponents $2, 1, 1, 2, 2, 1, 2$, which determines a score sequence corresponding to D .

We received that the four approximate algorithms together reconstruct each score set satisfying the condition $d_m \leq 8$.

Figure 1 shows the average running time of BALANCING, SHIFTENING, SHORTENING and HOLE as the function of m (the size of the score set).

5. ENUMERATION OF THE RECONSTRUCTED SCORE SEQUENCES

There are $2^{\binom{n}{2}}$ different labeled n -tournaments using the same n distinct labels, since for each pair of distinct labels $\{a, b\}$, either the vertex labeled a dominates the vertex labeled b or b dominates a [6, p. 197].

The following assertion characterizes the number $t(n)$ of non-isomorphic (unlabeled) n -tournaments.

Theorem 7. (Davis [2, 3, 8, 26]) *If $n \geq 1$, then*

$$(13) \quad t(n) \geq \frac{2^{\binom{n}{2}}}{n!},$$

n	D	BALANCING	SHORTENING	SHIFTENING
1	{7}	15	same	same
2	{6, 7}	7, 7	same	same
3	{5, 7}	9, 3	same	same
4	{5, 6, 7}	8, 2, 2	same	7, 2, 1
5	{4, 7}	6, 6	same	same
6	{4, 6, 7}	8, 1, 1	same	same
7	{4, 5, 7}	6, 2, 3	same	same
8	{4, 5, 6, 7}	6, 1, 2, 2	same	3, 6, 1, 1
9	{3, 7}	7, 1	same	same
10	{3, 6, 7}	3, 1, 9	same	same
11	{3, 5, 7}	5, 1, 5	same	same
12	{3, 5, 6, 7}	6, 1, 1, 1	same	same
13	{3, 4, 7}	6, 1, 2	same	same
14	{3, 4, 6, 7}	5, 2, 1, 1	same	same
15	{3, 4, 5, 7}	4, 3, 1, 1	same	same
16	{3, 4, 5, 6, 7}	2, 2, 3, 2, 2	same	same
17	{2, 7}	5, 5	same	4, 4
18	{2, 6, 7}	5, 2, 2	same	same
19	{2, 5, 7}	4, 1, 6	5, 1, 3	same
20	{2, 5, 6, 7}	5, 1, 1, 1	same	same
21	{2, 4, 7}	3, 4, 2	same	same
22	{2, 4, 6, 7}	1, 1, 5, 6	same	2, 1, 5, 1
23	{2, 4, 5, 7}	4, 1, 2, 2	same	same
24	{2, 4, 5, 6, 7}	3, 3, 1, 1, 1	same	same
25	{2, 3, 7}	3, 3, 3	same	same
26	{2, 3, 6, 7}	3, 3, 1, 1	same	3, 1, 2, 1
27	{2, 3, 5, 7}	2, 2, 2, 5	4, 1, 1, 1	same
28	{2, 3, 5, 6, 7}	1, 1, 1, 2, 8	same	same
29	{2, 3, 4, 7}	2, 2, 5, 2	3, 1, 3, 1	4, 1, 1, 3
30	{2, 3, 4, 6, 7}	1, 1, 1, 1, 9	4, 1, 1, 1, 1	same
31	{2, 3, 4, 5, 7}	1, 1, 1, 5, 3	3, 2, 1, 1, 1	same
32	{2, 3, 4, 5, 6, 7}	3, 2, 1, 1, 1	same	same

TABLE 1. Reconstruction results of the score sets beginning with the score $2 \leq d_1 \leq 7$ and ending with $d_m = 7$.

further

$$(14) \quad \lim_{n \rightarrow \infty} \frac{t(n)}{2^{\binom{n}{2}}/n!} = 1.$$

n	D	BALANCING	SHORTENING	SHIFTENING
1	{1, 7}	3, 9	same	same
2	{1, 6, 7}	3, 4, 4	same	same
3	{1, 5, 7}	3, 2, 6	same	same
4	{1, 5, 6, 7}	1, 1, 5, 6	same	3, 1, 5, 1
5	{1, 4, 7}	2, 2, 8	3, 3, 3	same
6	{1, 4, 6, 7}	1, 1, 4, 7	3, 3, 1, 1	same
7	{1, 4, 5, 7}	2, 2, 2, 5	3, 2, 2, 1	same
8	{1, 4, 5, 6, 7}	2, 4, 1, 1, 1	same	3, 1, 1, 2, 3
9	{1, 3, 7}	no	3, 1, 7	same
10	{1, 3, 6, 7}	1, 1, 3, 8	same	2, 2, 5, 1
11	{1, 3, 5, 7}	2, 2, 1, 6	3, 1, 3, 1	same
12	{1, 3, 5, 6, 7}	1, 1, 5, 2, 2	3, 1, 1, 1, 4	1, 1, 1, 6, 3
13	{1, 3, 4, 7}	1, 1, 1, 10	2, 1, 4, 1	3, 1, 1, 5
14	{1, 3, 4, 6, 7}	no	2, 3, 1, 1, 1	3, 1, 1, 2, 2
15	{1, 3, 4, 5, 7}	2, 2, 1, 2, 2	3, 1, 1, 1, 3	same
16	{1, 3, 4, 5, 6, 7}	no	3, 1, 1, 1, 1, 1	same
17	{1, 2, 7}	2, 2, 7	same	same
18	{1, 2, 6, 7}	2, 2, 3, 3	same	1, 3, 4, 2
19	{1, 2, 5, 7}	1, 1, 1, 10	2, 2, 3, 1	same
20	{1, 2, 5, 6, 7}	no	2, 2, 1, 1, 4	1, 3, 2, 2, 1
21	{1, 2, 4, 7}	no	2, 2, 1, 5	same
22	{1, 2, 4, 6, 7}	2, 2, 1, 2, 2	same	1, 3, 1, 3, 1
23	{1, 2, 4, 5, 7}	1, 1, 1, 4, 4	2, 2, 1, 1, 3	same
24	{1, 2, 4, 5, 6, 7}	2, 2, 1, 1, 1, 1	same	same
25	{1, 2, 3, 7}	no	2, 1, 2, 5	same
26	{1, 2, 3, 6, 7}	no	1, 1, 4, 1, 1	same
27	{1, 2, 3, 5, 7}	no	no	2, 1, 2, 1, 3
28	{1, 2, 3, 5, 6, 7}	1, 1, 1, 2, 3, 3	2, 1, 2, 1, 1, 1	same
29	{1, 2, 3, 4, 7}	1, 1, 1, 4, 2	1, 1, 2, 3, 1	same
30	{1, 2, 3, 4, 6, 7}	no	no	1, 1, 1, 2, 4, 1
31	{1, 2, 3, 4, 5, 7}	1, 1, 1, 1, 2, 5	1, 1, 3, 1, 1, 1	same
32	{1, 2, 3, 4, 5, 6, 7}	1, 1, 1, 3, 1, 1, 1	same	1, 1, 1, 1, 2, 3, 1

TABLE 2. Reconstruction results of score sets beginning with the score $d_1 = 1$ and ending with $d_m = 7$.

Proof. See [2, 3]. □

The concrete values of $t(n)$ can be found e.g. in [22] for $n = 1, 2, \dots, 76$. (13) gives not only an upper bound, but also a good approximation [5].

We enumerated the following cardinalities.

1) number of score sets $\sigma_m(n)$ belonging to the fixed number n of vertices and maximal number m of elements of D , further the distribution $\sigma_m(n, i)$, where $\sigma_m(n, i)$ gives the distribution of the number of score sets, containing i elements at fixed n and m ;

Table 3 contains $\sigma_m(n, i)$ and $\sigma_m(n)$ for $n = 1, \dots, 12$, $m = 12$, and $i = 1, \dots, 12$.

n, i	1	2	3	4	5	6	7	8	9	10	11	12	$\sigma_{12}(n)$
1	1	0	0	0	0	0	0	0	0	0	0	0	1
2	0	1	0	0	0	0	0	0	0	0	0	0	1
3	1	0	1	0	0	0	0	0	0	0	0	0	2
4	0	3	0	1	0	0	0	0	0	0	0	0	4
5	1	0	6	0	1	0	0	0	0	0	0	0	8
6	0	4	4	10	0	1	0	0	0	0	0	0	19
7	1	0	15	13	15	0	1	0	0	0	0	0	45
8	0	8	12	39	28	21	0	1	0	0	0	0	109
9	1	2	34	55	82	50	28	0	1	0	0	0	253
10	0	7	28	115	150	153	80	36	0	1	0	0	570
11	1	0	57	150	310	327	260	119	45	0	1	0	1270
12	0	13	60	262	502	705	622	412	168	55	0	1	2800

TABLE 3. Number of score sequences $\sigma_{12}(n, i)$ and $\sigma_{12}(n)$.

2) Number of score sequences $\tau_m(n)$ belonging to the fixed number n of vertices and fixed maximal number of the elements of D , further $\tau_m(n, i)$, where $\tau_m(n, i)$ gives the distribution of the number of score sequences, containing i elements at fixed n and m ;

Table 4 shows $\tau_m(n)$ and $\tau_m(n, i)$ for $n = 1, \dots, 12$, $m \leq n + 1$, and $i = 1, \dots, 12$.

3) Number of score sequences $\delta_{11}(n)$ belonging to fixed d_m and number n of vertices, further the distribution $\delta_{11}(n, i)$, where $\delta_{11}(n, i)$ gives the distribution the number of sequences, containing i elements at fixed $d_m = 11$ and n .

Table 5 shows the values of $\delta_{11}(n)$ and $\delta_{11}(n, i)$ for $n = 1, \dots, 12$, $d_m = 11$, and $i = 1, \dots, 11$.

6. SUMMARY

Checking all relevant score sets by polinomial time approximate algorithms we proved Theorem 5 for score sets whose maximal element is less than 9. Our proof is constructive, since we generated score sequences corresponding to the investigated score sets.

n, i	1	2	3	4	5	6	7	8	9	10	11	12	$\tau(n)$
1	1	0	0	0	0	0	0	0	0	0	0	0	1
2	0	1	0	0	0	0	0	0	0	0	0	0	1
3	1	0	1	0	0	0	0	0	0	0	0	0	2
4	0	3	0	1	0	0	0	0	0	0	0	0	4
5	1	0	7	0	1	0	0	0	0	0	0	0	9
6	0	4	4	13	0	1	0	0	0	0	0	0	22
7	1	0	21	14	22	0	1	0	0	0	0	0	59
8	0	8	14	70	40	34	0	1	0	0	0	0	167
9	1	2	46	96	204	90	50	0	1	0	0	0	490
10	0	7	34	267	414	511	182	70	0	1	0	0	1486
11	1	0	93	352	1200	1400	1165	332	95	0	1	0	4639
12	0	13	90	741	2252	4525	4068	2420	570	125	0	1	14805

TABLE 4. Number of score sequences $\tau_m(m)$ and $\tau(n)$.

n, i	0	1	2	3	4	5	6	7	8	9	10	11	$\delta_{11}(n)$
1	1	0	0	0	0	0	0	0	0	0	0	0	1
2	0	1	0	0	0	0	0	0	0	0	0	0	1
3	0	1	1	0	0	0	0	0	0	0	0	0	2
4	0	0	2	2	0	0	0	0	0	0	0	0	4
5	0	0	1	4	4	0	0	0	0	0	0	0	9
6	0	0	0	3	10	9	0	0	0	0	0	0	22
7	0	0	0	1	10	26	22	0	0	0	0	0	59
8	0	0	0	0	5	33	70	59	0	0	0	0	167
9	0	0	0	0	1	22	103	197	167	0	0	0	490
10	0	0	0	0	0	7	88	321	580	490	0	0	1486
11	0	0	0	0	0	1	43	329	1018	1762	1486	0	4639
12	0	0	0	0	0	0	11	213	1180	3280	5482	4639	14805

TABLE 5. Number of score sequences $\delta_{11}(n, i,)$ and $\delta_{11}(n)$.

The list of parameters and pseudocodes of the four approximating algorithms, further the generated score sequences can be found in [4, 5]:

<http://elekjani.web.elte.hu>.

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