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FIRST PRICE AND SECOND PRICE AUCTION GAMES. EQUILIBRIA DETECTION.

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Abstract. Three equilibria concepts - Nash, Aumann (strong Nash) and t-immune - are analyzed for first price and second price auction games. An evolutionary algorithm is used to detect these equilibria. Numerical experiments illustrate our assumptions regarding the equilibrium concepts.

1. Introduction

Game equilibrium detection is an important task in non-cooperative game theory. Equilibria may predict the outcome of games and can help decision makers (agents) to choose the "right" decision. Generally an equilibrium is a situation which is satisfactory for each player.

The most used equilibrium concept in non-cooperative game theory is the Nash equilibrium [10], however with some limitations:

- Nash equilibrium assumes that all players are rational - to choose a strategy which is favorable for another player - even if it would increase its own payoff - is not allowed, players follow the principle of unilateral maximization of their own payoff;
- in a game having several Nash equilibria a selection problem arises;

To solve these problems other equilibria were introduced, among which we mention:

- the t-immune strategies [1] capture the situation where agents are acting in an unpredictable manner; an irrational behaviour occurs in their choices;
- the strong Nash (Aumann) equilibrium [2], which is a refinement of the Nash equilibrium that can reduce the set of Nash equilibria of a certain game;

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The goal of this paper is to study these less known equilibria in a special class of games, in auction games as to the best of our knowledge only Nash equilibria was studied in auction games. First price and second price auction games have several Nash equilibria. The strong Nash and t-immune equilibria of an auction game can lead to unexpected results and can give some new interpretations for the auction model.

We use a computational tool, based on an evolutionary algorithm, to compute the Nash, strong Nash, and t-immune equilibria.

This paper has five sections, including this introductory section. The second section is concerned with the game theoretical prerequisites. Section three presents the evolutionary detection method. Section four describes numerical experiments. Finally, section five concludes the paper.

2. Game theoretical basic notions

2.1. **Game equilibria.** Mathematically, a finite strategic game is a system $G = (N, S_i, u_i, i = 1, n)$, where:

- N represents a set of players, and n is the number of players;
- for each player $i \in N$, S_i is the set of actions available,

$$S = S_1 \times S_2 \times ... \times S_n$$

  is the set of all possible situations of the game.

  Each $s \in S$ is a strategy (or strategy profile) of the game;
- for each player $i \in N$, $u_i : S \to R$ represents the payoff function of player i.

2.1.1. *Nash equilibrium.* The most popular and used equilibrium concept is the Nash equilibrium [10]. Playing in Nash sense means that no player can improve his payoff by deviating from its strategy only by himself.

Let us denote by (s_i, s^*_{-i}) the strategy profile obtained from s^* by replacing the strategy of player i with $s_i : (s_i, s^*_{-i}) = (s^*_1, ..., s_i, ..., s^*_n)$.

Definition 1 (Nash equilibrium). *A strategy profile $s^* \in S$ is a Nash equilibrium if*

$$u_i(s_i, s^*_{-i}) \leq u_i(s^*)$$

holds $\forall i = 1, ..., n, \forall s_i \in S_i$.

2.1.2. *Strong Nash (Aumann) equilibrium.* The Aumann (or strong Nash) equilibrium is a strategy for which no coalition of players has a joint deviation that improves the payoff of each member of the coalition.

Definition 2 (Strong Nash equilibrium)**.** *The strategy s^* is a Strong Nash (Aumann) equilibrium if $\forall I \subseteq N, I \neq \emptyset$ there does not exist any s_I such that the inequality*

$$u_i(s_I, s^*_{N-I}) > u_i(s^*)$$

holds $\forall i \in I$.

2.1.3. *t-immune equilibrium.* The t-immune equilibria [1] describes such a situations, where player act unpredictable. A strategy profile is t-immune when less then t player change but without decreasing the payoffs of the other players.

Definition 3. *A strategy $s^* \in S$ is a t-immune if for all $T \subseteq N$ with $card(T) \leq t$, all $s_T \in S_T$, and all $i \notin T$ we have:*

$$u_i(s^*_{-T}, s_T) \geq u_i(s^*).$$

2.2. **Auction games.** A considerable amount of literature has been published concerning auction games, out of which we mention [8], [12].

Auction games can be classified by different characteristics; based on the information knowledge we can distinguish complete and incomplete information games.

Complete information auction games [5] include: all-pay auctions, Amsterdam auctions, unique bid-auctions, open ascending-bid auctions (English auctions), descending-bid auctions (Dutch auctions). Numerical experiments will concern the following two auction types:

- first-price sealed bid auction - each bidder submits her/his own bid without seeing other bids, and the object is sold to the highest bidder at her/his bid, who pays her/his own bid;
- second-price sealed bid auction (Vickrey auctions)- each bidder submits her/his own bid, the object is sold to the highest bidder, who pays the second highest price for the object;

3. Evolutionary equilibrium detection

The equilibrium detection problem is similar to a multiobjective optimization problem (MOP), where the commonly accepted solution is the Pareto optimal set. Potential solutions of a MOP are compared by using the Pareto dominance relation. In a similar manner, different equilibria types can be computed by using different generative relations that will guide the search toward certain equilibria.

3.1. **Generative relations.** In what follows the generative relations for the Nash, Aummann and t-immune equilibria are presented.

3.1.1. *Generative relation for Nash equilibrium.* Such a relation has been defined for Nash equilibria in [9] by using a quality measure $k(s, q)$ denoting the number of players that benefit from unilaterally switching their choices from s to q:

$$k(s^*, s) = card\{i \in N, u_i(s_i, s^*_{-i}) > u_i(s^*), s_i \neq s_i\},$$

where $card\{M\}$ denotes the cardinality of the set M.

Definition 4. *Let $q, s \in S$. We say the strategy q is better than strategy s with respect to Nash equilibrium (q Nash ascends s, and we write $q \prec_N s$, if the following inequality holds:*

$$k(q, s) < k(s, q).$$

Definition 5. *The strategy profile $q \in S$ is called Nash non-dominated, if and only if there is no strategy $s \in S, s \neq q$ such that*

$$s \prec_N q.$$

The relation $\prec_N$ is a generative relation for Nash equilibrium in the sense that the set of non-dominated strategies with respect to $\prec_N$ is equal to the set of Nash equilibria [9].

3.1.2. *Generative relation for strong Nash equilibrium.* A relative quality measure of two strategies with respect to Aumann equilibrium can be defined as [4], [6]:

$$a(s^*, s) = card[i \in I, \emptyset \neq I \subseteq N, u_i(s_I, s^*_{N-I}) > u_i(s^*), s_i \neq s^*_i],$$

where $card[M]$ denotes the cardinality of the multiset M (an element i can appear several times in M and each occurrence is counted in $card[M]$). Thus, $a(s^*, s)$ counts the total number of players that would benefit from collectively switching their strategies from s^* to s.

Definition 6. *Let $s^*, s \in S$. We say the strategy s^* is better than strategy s with respect to Aumann equilibrium, and we write $s^* \prec_A s$, if the following inequality holds:*

$$a(s^*, s) < a(s, s^*).$$

Definition 7. *The strategy profile $s^* \in S$ is called Aumann non-dominated, if and only if there is no strategy $s \in S, s \neq s^*$ such that*

$$s \prec_A s^*.$$

The relation $\prec_A$ can be considered as the generative relation for Aumann equilibrium, i.e. the set of non-dominated strategies, with respect to $\prec_A$, induces the *Aumann equilibrium*.

3.1.3. *Generative relation for t-immune strategies.* Consider a quality measure $t(s^*, s)$, which denotes the number of players who gain by switching from one strategy to the other strategies [7]:

$$t(s^*, s) = card[i \in N - T, u_i(s_T, s^*_{-T}) \leq u_i(s^*), s_T \neq s^*_T, card(T) = t, T \subseteq N],$$

where $card[M]$ represents the cardinality of the set M.

Definition 8. *Let $s^*, s \in S$. We say the strategy s^* is better than strategy s with respect to t-immunity, and we write $s^* \prec_T s$, if the following inequality holds:*

$$t(s^*, s) < t(s, s^*).$$

Definition 9. *The strategy profile $s^* \in S$ is called t-immune non-dominated, if and only if there is no strategy $s \in S, s \neq s^*$ such that*

$$s \prec_T s^*.$$

The relation $\prec_T$ can be considered as the generative relation for t-immune equilibrium, i.e. the set of non-dominated strategies, with respect to $\prec_T$, induces the *t-immune strategies.*

3.2. **Evolutionary detection method.** For evolutionary equilibrium detection the Relational Evolutionary Equilibrium Detection Method (REED) is used. REED is based on NSGA2 [3] with the only difference that the Pareto domination relation has been replaced with the appropriate generative relation.

REED can be described as follows:

REED method

S1. Set $t = 0$;
S2. Randomly initialize a population $P(0)$ of strategy profiles;
- Repeat until the maximum generation number is reached:

S3. Binary tournament selection and recombination using the simulated binary crossover (SBX) operator for $P(t) \to Q$;
S4. Mutation on Q using real polynomial mutation $\to P$;
S5. Compute the rank of each population member in $P(t) \cup P$ with respect to the generative relation (Nash, Aumman, t-immune). Order by rank $(P(t) \cup P)$;
S6. Rank based selection for survival $\to P(t + 1)$;

4. Experiments

Parameter settings for the numerical experiments are presented in Table 1.

TABLE 1. Parameter settings for the evolutionary algorithm used for the numerical experiments

Parameter	
Population size	100
Max. no. of generations	100
prob. of crossover	0.2
prob. of mutation	0.2

4.1. **Experiment 1 - First-price sealed bid auction.** In the first-price sealed bid auction two players cast their bid independently. The value of the bidding objects is v_1 for the first player and v_2 for the second one. The winner is the highest bidder, who needs to pay his own bid. A simple agreement is specified in case of a tie: if both have equal bids, the winner is the first bidder (another variant is to randomly choose a winner).

The payoff functions are the following [11]:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1, & \text{if } b_1 \geq b_2, \\ 0, & \text{otherwise.} \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_2, & \text{if } b_2 > b_1, \\ 0, & \text{otherwise.} \end{cases}$$

The game has several Nash equilibria as all $v_1 \leq b_1^* = b_2^* \leq v_2$ is a Nash equilibrium of the game. Aumann equilibrium is a refinement of the Nash equilibrium, therefore reduces the set of Nash equilibria. t-immunity gives a perturbation of a game, describes a situation in which players act irrational.

Figure 1 shows the evolutionary detected t-immune, Nash and Aumann equilibria of the game for $v_1 = 5$ and $v_2 = 4$. Figure 2 presents the same equilibria for $v_1 = 3$ and $v_2 = 5$.

In both cases ($v_1 = 5$,$v_2 = 4$; $v_1 = 3$, $v_2 = 5$) Aumann equilibrium is the most "favorable" equilibrium: the strategy profile $(4, 4)$ with the corresponding payoff $(1, 0)$ for the first case, and for the second one is $(3, 3)$ with the corresponding payoff $(0, 2)$. t-immune equilibrium in both cases proves the irrational behavior of the players: nobody has a positive gain, one of the players has a payoff of 0, and the other one has a negative payoff.

4.2. **Experiment 2 - Second-price sealed bid auction.** In a second-price auction game the winner needs to pay the second highest bid. In the two player version of the model the value of the bidding objects is v_1 for the first player and v_2 for the second one [11]:

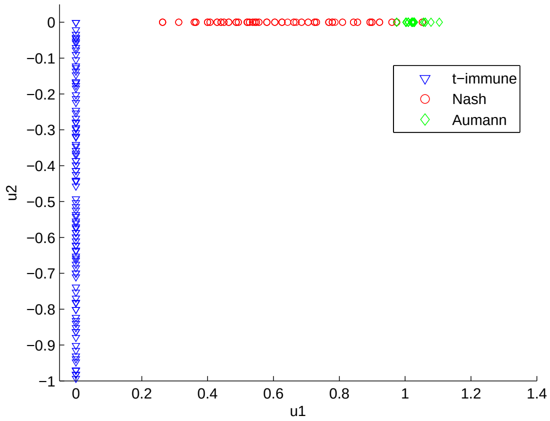

FIGURE 1. G_1. Detected t-immune, Nash and Aumann payoffs for the first price auction games, $v_1 = 5$ and $v_2 = 4$

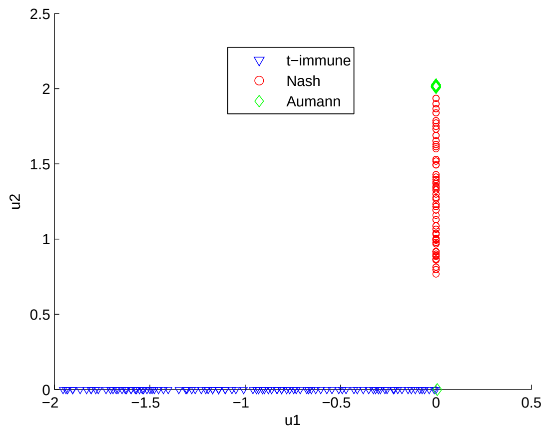

FIGURE 2. G_1. Detected t-immune, Nash and Aumann payoffs for the first price auction games,$v_1 = 3$ and $v_2 = 5$

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2, & \text{if } b_1 \geq b_2, \\ 0, & \text{otherwise.} \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_1, & \text{if } b_2 > b_1, \\ 0, & \text{otherwise.} \end{cases}$$

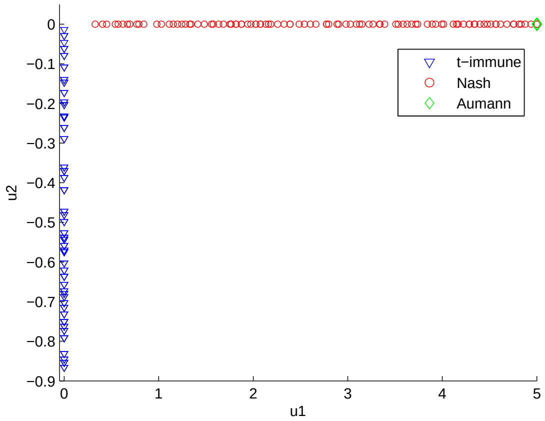

FIGURE 3. G_1. Detected t-immune, Nash and Aumann payoffs for the second price auction games, $v_1 = 5$ and $v_2 = 4$

In the second-price sealed bid auction also exist multiple Nash equilibria [11]: every strategy profile $(b_1, b_2) = (v_1, v_2)$, $(b_1, b_2) = (v_1, 0)$, or $(b_1, b_2) = (v_2, v_1)$, etc.

Figure 3 presents the evolutionary detected t-immune, Nash and Aumann equilibria of the game for $v_1 = 5$ and $v_2 = 4$. Figure 4 depicts the evolutionary detected t-immune, Nash and Aumann equilibria for $v_1 = 3$ and $v_2 = 5$.

In this case the Aumann equilibrium refines the set of Nash equilibria: the payoff $(5, 0)$ is the unique Aumann payoff in the first case ($v_1 = 5$ and $v_2 = 4$) and $(3, 0)$, $(0, 5)$ in the second case ($v_1 = 3$ and $v_2 = 5$). Regarding to the t-immune equilibrium, we can notice the same results as in the case of the first-price auction game: nobody has a positive gain, one of the players has a payoff of 0, and the other one has a negative payoff (for both cases).

5. CONCLUSIONS

First price and second price auction games are analyzed using different equilibrium concepts. Nash equilibrium is the standard equilibrium concept, but we focus on a refinement of the Nash equilibrium: the Aumann (strong Nash) equilibrium, and on the t-immune equilibrium, which can model irrational behavior. An evolutionary algorithm, based on generative relations, is used in order to detect these equilibria. Numerical experiments illustrate our assumptions regarding the three studied equilibria.

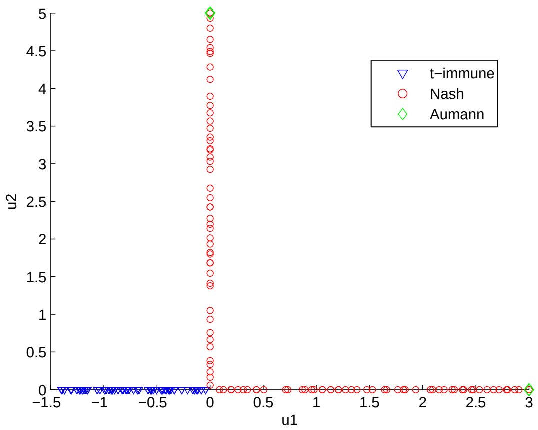

Figure 4. G_1. Detected t-immune, Nash and Aumann payoffs for the second price auction games, $v_1 = 3$ and $v_2 = 5$

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