A GOOD DRAWING OF COMPLETE BIPARTITE GRAPH $K_{9,9}$, WHOSE CROSSING NUMBER HOLDS ZARANKIEWICZ CONJECTURES

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ABSTRACT. There exist some Drawing for any graph $G = (V, E)$ on plan. An important aim in Graph Theory and Computer science is obtained a best drawing of an arbitrary graph. Also, a draw of a non-planar graph G on plan generate several edge-cross. A good drawing (or strongly best drawing) of G is consist of minimum edge-cross.

The crossing number of a graph G, is the minimum number of crossings in a drawing of G in the plane, denoted by $cr(G)$. A crossing is a point of intersection between two edges. The crossing number of the complete bipartite graph is one of the oldest crossing number open problems.

In this paper, we present a good drawing of complete bipartite graph $K_{9,9}$. This drawing is able to developed on $K_{n,n}$, $\forall n \leq 9$ and implies that the crossing number of these graphs hold Zarankiewicz conjecture. $\forall n, m \in \mathbb{N}$ Zarankiewicz conjecture is equal to

$$cr(K_{n,m}) \stackrel{?}{=} Z(m, n) = [\frac{m}{2}][\frac{m-1}{2}][\frac{n}{2}][\frac{n-1}{2}].$$

1. INTRODUCTION

Let $G = (V, E)$ be a simple finite connected graph with the vertex set $V(G)$ and the edge set $E(G)$. $|V(G)| = n$, $|E(G)| = e$ are the number of vertices and edges.

For each vertex v of a graph G, let $N_G(v) := \{u \in V(G) | uv \in E(G)\}$ be the neighborhood of v in G. The degree of v, denoted by $deg(v)$, is $|N_G(v)|$. Let $\Delta(G)$ be the maximum degree of a vertex of G.

The crossing number of a graph G, denoted by $cr(G)$, is the minimum number of crossings in a drawing of G in the plane.

A drawing of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such

that the only vertices an edge intersects are its own endpoints, and no three
edges intersect at a common point (except at a common endpoint). A drawing
is convex if in addition the vertices are in convex position. A crossing is a point
of intersection between two edges (other than a common endpoint). A drawing
with no crossings is crossing-free. A graph is planar if it has a crossing-free
drawing, see [4, 12, 22] for surveys. For example look at Figure 1, (planar
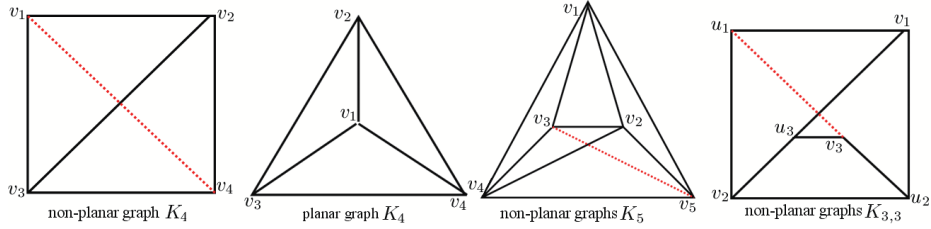graph K_4 and non-planar graphs K_5, $K_{3,3}$).

FIGURE 1. Figures of K_4, K_5 and $K_{3,3}$ on the plan

The crossing number is an important measure of the non-planarity of a
graph [18]. Computing the crossing number is NP-hard [5], and remains so for
simple cubic graphs [9, 13]. Moreover, the exact or even asymptotic crossing
number is not known for specific graph families, such as complete graphs [14],
complete bipartite graphs [11,14,16] and Cartesian products [1, 2, 6-8, 10, 15,
16, 19-21, 23, 24].

Determining the crossing number of the complete bipartite graph is one
of the oldest crossing number open problems. It was first posed by Turan and
known as Turan's brick factory problem. In 1954, Zarankiewicz conjectured
[24] that it is equal to

$$cr(K_{n,m}) \overset{?}{=} Z(m,n) = [\frac{m}{2}][\frac{m-1}{2}][\frac{n}{2}][\frac{n-1}{2}].$$

He even gave a proof and a drawing that matches the lower bound, but the
proof was shown to be flawed by Richard Guy [7]. Then in 1970 D.J. Kleitman
proved that Zarankiewicz conjecture holds for $Min(m;n) \leq 6$ [10]. In 1993
D.R. Woodall proved it for $m \leq 8$; $n \leq 10$ [23]. Previously the best known
lower bound in the general case for all $m, n \in \mathbb{N}$ was the one proved by D.J.
Kleitman [10]:

$$cr(K_{n,m}) \geq \frac{1}{5}(m(m-1))[\frac{n}{2}][\frac{n-1}{2}].$$

Now, we have the better lower bound [11]

$$cr(K_{n,m}) \geq \frac{1}{5}(m(m-1))[\frac{n}{2}][\frac{n-1}{2}] + 9.9 \times 10^{-6}m^2n^2.$$

for sufficiently large m and n.

Upper bounds on the crossing number of general families of graphs have been less studied. Obviously $cr(K_{n,m}) \leq \binom{|E(G)|}{2}$ for every graph G.

2. Drawing of complete Bipartite graph $K_{9,9}$

D.R. Woodall [10] used an elaborate computer search to show that Zarankiewicz conjecture holds for $K_{7,7}$ and $K_{7,9}$. Thus, one of the smallest unsettled instance of Zarankiewicz conjecture is $K_{9,9}$. For further research see paper series [8, 10, 11, 17-21].

So, we focus on the best drawing of complete bipartite graph $K_{9,9}$ and compute its crossing number for this drawing. In continue, we claim that this drawing is a best drawing for $K_{9,9}$ and $cr_D(K_{9,9})$ hold Zarankiewicz conjecture. By according the Figure 5. Also we show that by similar drawing for $K_{7,7}$ which is a best drawing of it and hold Zarankiewicz conjecture, it is maybe another proof of $cr_D(K_{7,7})$.

Before beginning present of this drawing, we give some definitions that will be used throughout the paper.

Definition 1. The crossing number $cr(G)$ of a graph G is the smallest crossing number of any drawing of G in the plane, where the crossing number cr of a drawing D is the number of non-adjacent edges that have a crossing in the drawing.

Definition 2. A good drawing a graph G is a drawing where the edges are non-self-intersecting where each two edges have at most one point in common, which is either a common end vertex or a crossing.

Clearly a drawing with minimum crossing number must be a good drawing (or for strongly a best drawing) and obviously a good drawing of planar graph G is the crossing-free drawing.

Definition 3. Suppose $V = \{v_1, v_2, ..., v_n\}$ is the vertex set of an arbitrary graph G. Then $E(G)$ (the edge set of G) is consist of $e_{i,j}$, such that v_i is adjacent with v_j ($\forall i, j \in \mathbb{Z}_n = \{1, 2, ..., n\}$). Now, Pair-Cross Matrix of G ($CR(G) = [cr_{i,j}]_{i,j \in \mathbb{Z}_n=}$) presents the number of all cross on the edge $e_{i,j}$.

It's obvious that, if v_i, v_j be the non-adjacent vertices, then $cr_{i,j} = 0$. Since, there exist many different drawing for a graph G, therefore we have a Pair-Cross Matrix $CR_D(G)$ for any drawing D of G. Also, it's obvious that all Pair-Cross Matrix $CR_D(G)$ are symmetric and the members on the original diameter are equal to zero.

$$CR_D(G) = \begin{bmatrix} 0 & cr_{1,2} & cr_{1,3} & . & . & . & cr_{1,n} \\ cr_{2,1} & 0 & cr_{2,3} & . & . & . & cr_{2,n} \\ cr_{3,1} & cr_{3,2} & 0 & . & . & . & cr_{3,n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ cr_{n,1} & cr_{n,2} & cr_{n,3} & . & . & . & 0 \end{bmatrix}_{n \times n} \begin{array}{l} \to cr_{v_1} \\ \to cr_{v_2} \\ \to cr_{v_3} \\ . \\ . \\ . \\ \to cr_{v_n} \end{array}$$

$$\begin{array}{cccccc} & \downarrow & \downarrow & \downarrow & & \downarrow \\ (1) & cr_{u_1} & cr_{u_2} & cr_{u_3} & . & . & . & cr_{u_n} \end{array}$$

Example 1. By according to Figure 1, we see that the drawing D_1 is the crossing-free drawing of K_4. So Pair-Cross Matrix of K_4 will be equal to

$$CR_{D_1}(K_4) = 0$$

and also

$$CR_{D_2}(K_4) = \begin{array}{c} v_1 \to \\ v_2 \to \\ v_3 \to \\ v_4 \to \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{4 \times 4}$$

$$\begin{array}{cccc} & \uparrow & \uparrow & \uparrow & \uparrow \\ (2) & v_1 & v_2 & v_3 & v_4 \end{array}$$

Example 2. Similar Above (see Figure 1), Pair-Cross Matrix of $K_5, K_{3,3}$ on the best drawing D will be equal to

$$CR_{D_3}(K_5) = \begin{array}{c} v_1 \to \\ v_2 \to \\ v_3 \to \\ v_4 \to \\ v_5 \to \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{5 \times 5}$$

$$\begin{array}{ccccc} (3) & v_1 & v_2 & v_3 & v_4 & v_5 \end{array}$$

and

$$CR_{D_4}(K_{3,3}) = \begin{array}{c} v_1 \to \\ v_2 \to \\ v_3 \to \\ u_1 \to \\ u_2 \to \\ u_3 \to \end{array} \begin{bmatrix} 0 & 0 & 0 & |0 & 0 & 1 \\ 0 & 0 & 0 & |0 & 0 & 0 \\ 0 & 0 & 0 & |\underline{1} & \underline{0} & \underline{0} \\ \overline{0} & \overline{0} & \overline{1}| & 0 & 0 & 0 \\ 0 & 0 & 0| & 0 & 0 & 0 \\ 1 & 0 & 0| & 0 & 0 & 0 \end{bmatrix}_{6 \times 6}$$

$$\begin{array}{cccccc} (4) & v_1 & v_2 & v_3 & u_1 & u_2 & u_3 \end{array}$$

Corollary 1. The summation of all members of $CR_D(G)$ implies that is equal to the crossing number $CR_D(G)$ of a graph G on the drawing D. In other words

$$(5) \qquad CR_D(G) = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} cr_{i,j} = \frac{\sum_{i=1}^{n} cr_{v_i}}{4}$$

Definition 4. Let $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{u_1, u_2, ..., u_m\}$ be two partitions of $V(K_{m,n})$, where $V(K_{m,n})$ is the vertex set of the complete bipartite graph $K_{m,n}$. Now, the Pair-Cross Matrix $CR_D^*(K_{m,n})$ presents the number of all cross on the edge $e_{i,j} = v_i u_j$ as follow:

$$(6) \qquad CR_D^*(K_{m,n}) = V_1 \{ \overbrace{\left[cr_{v_i u_j} \right]}^{V_2} \}_{n \times m}$$

We redefine this matrix for $K_{m,n}$, because by rewrite Definition 3 for $G = K_{m,n}$ then

$$CR_D(K_{m,n}) = \begin{matrix} V_1 \rightarrow \\ V_2 \rightarrow \end{matrix} \quad \left[\begin{matrix} 0 & CR_D^*(K_{m,n}) \\ CR_D^*(K_{m,n})^t & 0 \end{matrix} \right]_{(m+n) \times (m+n)}$$

$$(7) \qquad\qquad\qquad\qquad V_1 \qquad\qquad\qquad V_2$$

and $CR_D^*(K_{m,n}) = CR_D^*(K_{m,n})^t$.

Example 3. By according to Figure 1, it is obvious that modified Pair-Cross Matrix of $K_{3,3}$ is

$$(8) \qquad CR_D^*(K_{3,3}) = \left[\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \right]$$

Corollary 2. The summation of all members of $CR_D^*(K_{m,n})$ is equal to the crossing number $CR_D(K_{m,n})$ of a complete bipartite graph on the drawing D. Thus

$$(9) \qquad CR_D(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} cr_{i,j}^*.$$

2.1. **Method.** In this subsection, we achieve a good drawing D of complete bipartite graph $K_{9,9}$ and conclude the crossing number $CR_D(K_{9,9})$ of it. We start this process with an arbitrary drawing D of complete bipartite graph $K_{9,9}$ and we make Pair-Cross Matrix $CR_D^*(K_{9,9})$ by according to the drawing D. So, we find a large member of $CR_D^*(K_{9,9})$ ($\Delta cr = Max\{cr_{i,j}^* | i, j \in \mathbb{Z}_9\}$) and we redraw the correlate edge with it, such that decrease the number of

cross on the correlate edge. Notice that this change must n't many increase other members of $CR_D^*(K_{9,9})$. By repeat this process several times, upshot we will have a good drawing of complete bipartite graph $K_{9,9}$. See Figure 3 (For look Figure 3, attention to Appendix 1.) and Pair-Cross Matrix $CR_D^*(K_{9,9})$ is equal to

$$
CR_D^*(K_{9,9}) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{array}
\begin{array}{ccccccccc}
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\
\left(\begin{array}{ccccc|cccc}
16 & 12 & 8 & 4 & 0 & 0 & 4 & 8 & 12 \\
12 & 9 & 6 & 3 & 0 & 0 & 3 & 6 & 9 \\
8 & 6 & 5 & 4 & 3 & 4 & 5 & 6 & 7 \\
4 & 3 & 4 & 5 & 6 & 8 & 7 & 6 & 5 \\
0 & 0 & 3 & 6 & 9 & 12 & 9 & 6 & 3 \\
0 & 0 & 4 & 8 & 12 & 16 & 12 & 8 & 4 \\
4 & 3 & 5 & 7 & 9 & 12 & 10 & 8 & 6 \\
8 & 6 & 6 & 6 & 6 & 8 & 8 & 8 & 8 \\
12 & 9 & 7 & 5 & 3 & 4 & 6 & 8 & 10
\end{array} \right)
\end{array}
\begin{array}{l}
\longrightarrow 64 = 4(4)^2 \\
\longrightarrow 48 = 3(4)^2 \\
\longrightarrow 48 = 3(4)^2 \\
\longrightarrow 48 = 3(4)^2 \\
\longrightarrow 48 = 3(4)^2 \\
\longrightarrow 64 = 4(4)^2 \\
\longrightarrow 64 = 4(4)^2 \\
\longrightarrow 64 = 4(4)^2 \\
\longrightarrow_+ 64 = 4(4)^2
\end{array}
$$

$$\left(\frac{512}{2} \right) = 256 = (4)^2 (4)^2$$

By refer to Figure 3 of complete bipartite graph $K_{9,9}$, it is obvious to see that this figure is symmetric (near to symmetric) and the vertices are in the two opponent cycles (eight vertices as a common set in one of cycles and one remaining vertex is a center of another cycle). As well as, these two cycles and their covered vertices have stated in a mirror (See close up view of $K_{9,9}$ in Figure 2).

FIGURE 2. The close up view of $K_{9,9}$ with two cycles that covered vertices (black and red cycles).

Now, by according to the matrix $CR_D^*(K_{9,9})$ and Figure 3, if we redraw an edge e_{uv}, then we increase the crossing number cr_{uv}^*) obviously. But, an important point is number 4 and its multiples in the matrix $CR_D^*(K_{9,9})$. Number

4 is important, since $4 = \left[\frac{9}{2}\right]$. On the other hand, number 3 is important in the matrix $CR^*_D(K_{7,7})$, since $3 = \left[\frac{7}{2}\right]$ similarly. See complete bipartite graph $K_{7,7}$ in Figure 4 (on Appendix 2) and Pair-Cross Matrix $CR^*_D(K_{7,7})$ as follow:

$$
CR^*_D(K_{7,7}) =
\begin{array}{c}
\\
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc}
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7
\end{array} \\
\left(
\begin{array}{ccc|cccc}
9 & 6 & 3 & 0 & 0 & 3 & 6 \\
6 & 4 & 2 & 0 & 0 & 2 & 4 \\
3 & 2 & 2 & 2 & 3 & 3 & 3 \\
0 & 0 & 2 & 4 & 6 & 4 & 2 \\
0 & 0 & 3 & 6 & 9 & 6 & 3 \\
3 & 2 & 3 & 4 & 6 & 5 & 4 \\
6 & 4 & 3 & 2 & 3 & 4 & 5
\end{array}
\right)
\end{array}
\begin{array}{l}
\longrightarrow \quad 27 = 3(3)^2 \\
\longrightarrow \quad 18 = 2(3)^2 \\
\longrightarrow \quad 18 = 2(3)^2 \\
\longrightarrow \quad 18 = 2(3)^2 \\
\longrightarrow \quad 27 = 3(3)^2 \\
\longrightarrow \quad 27 = 3(3)^2 \\
\longrightarrow_+ \, 27 = 3(3)^2 \\
\hline
\left(\frac{162}{2}\right) = 81 = (3)^2(3)^2
\end{array}
$$

3. Conclusions

In this report, we drawing $K_{9,9}$ in the plan with 256 crossing number. We obtained this drawing by draw $K_{9,9}$ step to step, such that we choose a large Cr on the Pair-Cross Matrix and redraw it for decrease crossing number. In fact, this work is quite tentative and experience, in other words, is handwork. In other way, we can drawing $K_{9,9}$ by add two vertices to a best drawing $K_{8,8}$ (Readers know that this graph have 144 crossing points in best drawing or $Cr(K_{8,8}) = 144$), and also we can obtain a best drawing $K_{8,8}$ by add two vertices to best drawing $K_{7,7}$ ($Cr(K_{7,7}) = 81$). In other words, For $h = 3, ..., 9$; we can draw all complete graphs $K_{h,h}$, that the crossing number of them hold Zarankiewicz conjecture.

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