

A PRIMAL-DUAL INTERIOR POINT ALGORITHM FOR CONVEX QUADRATIC PROGRAMS

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ABSTRACT. In this paper, we propose a feasible primal-dual path-following algorithm for convex quadratic programs. At each interior-point iteration the algorithm uses a full-Newton step and a suitable proximity measure for tracing approximately the central path. We show that the short-step algorithm has the best known iteration bound, namely $O(\sqrt{n} \log \frac{(n+1)}{\epsilon})$.

1. INTRODUCTION

Consider the quadratic program (PQ) in standard format:

$$(P) \min_x \left\{ c^T x + \frac{1}{2} x^T Q x : Ax = b, x \geq 0 \right\}$$

and its dual problem

$$(D) \max_{(x, y, z)} \left\{ b^T y - \frac{1}{2} x^T Q x : A^T y + z - Qx = c, z \geq 0 \right\}.$$

Here Q is a given $n \times n$ real symmetric matrix, A is a given $m \times n$ real matrix, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The vectors x, y, z are the vectors of variables.

Quadratic programs appear in many areas of applications. For example in finance (portfolio optimization) and also as subproblems in sequential Quadratic Programming [1,5,6,10,11]. Interior point methods (IPMs) are among the most effective methods to solve a large wide of optimization problems. Nowadays most popular and robust methods of them are primal-dual path-following algorithms due to their numerical efficiency and their theoretical polynomial complexity [1,2,6-11]. The success of these algorithms for solving linear optimization LO leads researchers to extend it naturally to other important problems such as semidefinite programs SDP, quadratic programs QP, complementarity CP and conic optimization problems COP.

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In this paper, we propose a feasible short-step primal-dual interior point (IP) algorithm for convex quadratic programs. The algorithm uses at each interior-point iteration a full-Newton step and a suitable proximity measure for tracing approximately the central path. For its complexity analysis, we reconsider the analysis used by many researchers for LO and we make it suitable for CQP case. We show that the algorithm has the best known iteration bound $O(\sqrt{n} \log \frac{(n+1)}{\epsilon})$ which is analogous to LO.

The rest of the paper is organized as follows. In section 2, a feasible short-step primal-dual IP algorithm for CQP is presented. In section 3, the complexity analysis of the algorithm is discussed. In section 4, a conclusion is stated.

The notation used in this paper is: \mathbb{R}^n denotes the space of n -dimensional real vectors. Given $x, y \in \mathbb{R}^n$, $x^T y = \sum_{i=1}^n x_i y_i$ is their usual scalar product whereas $xy = (x_1 y_1, \dots, x_n y_n)^T$ is the vector of their coordinate-wise product. The standard 2-norm and the maximum norm for a vector x are denoted by $\|x\|$ and $\|x\|_\infty$, respectively. Let $x \in \mathbb{R}^n$, $\sqrt{x} = (\sqrt{x_1}, \dots, \sqrt{x_n})^T$, $x^{-1} = (x_1^{-1}, \dots, x_n^{-1})^T$ if $x_i \neq 0$ for all i and $(\frac{x}{y}) = (\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n})^T$ for $y_i \neq 0$. Let $g(t)$ and $f(t)$ be two positive real valued functions, then $g(t) = O(f(t)) \Leftrightarrow g(t) \leq c f(t)$ for some $c > 0$. The vector of ones in \mathbb{R}^n is denoted by e .

2. A PRIMAL-DUAL PATH FOLLOWING IP ALGORITHM FOR CQP

Throughout the paper, we make the following assumptions on (P) and (D) .

- The matrix A is of rank m ($\text{rg}(A) = m < n$).
- Interior-Point-Condition (IPC). There exists (x^0, y^0, z^0) such that

$$Ax^0 = b, A^T y^0 - Qx^0 + z^0 = c, x^0 > 0, z^0 > 0.$$

- The matrix Q is positive semidefinite, i.e., for all $x \in \mathbb{R}^n : x^T Q x \geq 0$.

It is well-known under our assumptions that solving (P) and (D) is equivalent to solve the Karush-Khun-Tucker optimality conditions for (P) and (D) :

$$(1) \quad \begin{cases} Ax & = b, x \geq 0, \\ A^T y + z - Qx & = c, z \geq 0, \\ xz & = 0. \end{cases}$$

Now, by replacing the complementarity equation $xz = 0$ in (1) by the perturbed equation $xz = \mu e$, one obtains the following perturbed system:

$$(2) \quad \begin{cases} Ax & = b, x \geq 0, \\ A^T y + z - Qx & = c, z \geq 0, \\ xz & = \mu e, \end{cases}$$

with $\mu > 0$. It is also known under our assumptions that the system (2) has a unique solution for each $\mu > 0$, denoted by $(x(\mu), y(\mu), z(\mu))$, we call $x(\mu)$ the μ -center of (P) and $(y(\mu), z(\mu))$ the μ -center of (D) . The set of μ -centers

gives a homotopy path, which is called the central path of (P) and (D) . If μ goes to zero, then the limit of the central path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for (P) and (D) . The notion of the central path has been studied by many authors (see the books, e.g. [9] [10] and [11]).

Primal-dual path-following interior point algorithms are iterative methods which aim to trace approximately the central path by using at each interior iteration a feasible Newton step and get closer to a solution of (2) as μ goes to zero, e.g. [6-11].

Now, we proceed to describe a full-Newton step produced by the algorithm for a given $\mu > 0$. Applying Newton's method for (2) for a given feasible point (x, y, z) then the Newton direction $(\Delta x, \Delta y, \Delta z)$ at this point is the unique solution of the following linear system of equations:

$$\begin{cases} A\Delta x & = 0, \\ A^T\Delta y + \Delta z - Q\Delta x & = 0, \\ x\Delta z + z\Delta x & = \mu e - xz. \end{cases}$$

This last can be written as:

$$(3) \quad \begin{pmatrix} A & 0 & 0 \\ -Q & A^T & I \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu e - Xz \end{pmatrix},$$

where $X := \text{diag}(x)$, $Z := \text{diag}(z)$ and I is the identity matrix of order n . The format in (3) is suitable for numerical implementation. Hence an update full-Newton step is given by $x_+ = x + \Delta x$, $y_+ = y + \Delta y$ and $z_+ = z + \Delta z$. Now, we introduce a norm-based proximity as:

$$\delta(xz; \mu) = \frac{1}{2} \left\| \sqrt{\left(\frac{xz}{\mu}\right)^{-1}} - \sqrt{\frac{xz}{\mu}} \right\|,$$

to measure the closeness of feasible points to the central path. We use also a threshold value β and we suppose that a strictly feasible point (x^0, y^0, z^0) such that $\delta(x^0 z^0; \mu^0) \leq \beta$ for certain μ^0 , is known. The details of the algorithm are stated in the next subsection.

2.1. Algorithm.

<p>Input An accuracy parameter $\epsilon > 0$; a threshold parameter $0 < \beta < 1$ (default $\beta = \frac{1}{\sqrt{2}}$); a fixed barrier update parameter $0 < \theta < 1$ (default $\theta = \frac{1}{2\sqrt{n}}$); a strictly feasible (x^0, y^0, z^0) and μ^0 such that $\delta(x^0 z^0; \mu^0) \leq \beta$;</p> <p>begin $x := x^0; y := y^0; z := z^0; \mu := \mu^0$; while $n\mu \geq \epsilon$ do begin $\mu := (1 - \theta)\mu$; solve the system (3) to obtain: $(\Delta x, \Delta y, \Delta z)$; update $x = x + \Delta x, y = y + \Delta y, z = z + \Delta z$; end end</p>

Fig.1. Algorithm 2.1

One distinguishes IPMs as short-step when $\theta = O\left(\frac{1}{\sqrt{n}}\right)$ and large-step when $\theta = O(1)$.

3. COMPLEXITY ANALYSIS

In this section, we discuss the complexity analysis of Algorithm 2.1. For convenience, we introduce the following notation. The vectors

$$v := \sqrt{\frac{xz}{\mu}}, \quad d := \sqrt{\frac{x}{z}}.$$

The vector d is used to scale the vectors x and z to the same vector v :

$$\frac{d^{-1}x}{\sqrt{\mu}} = \frac{dz}{\sqrt{\mu}} = v,$$

as well as for the original directions Δx and Δz to

$$d_x := \frac{d^{-1}\Delta x}{\sqrt{\mu}}, \quad d_z := \frac{d\Delta z}{\sqrt{\mu}}.$$

In addition, we have

$$(4) \quad x\Delta z + z\Delta x = \mu v(d_x + d_z),$$

and

$$(5) \quad \Delta x\Delta z = \mu d_x d_z.$$

By using these notations the linear system in (3) and the proximity become:

$$(6) \quad \begin{cases} \bar{A} d_x = 0 \\ \bar{A}^T \Delta y + d_z - \bar{Q} d_x = 0 \\ d_x + d_z = v^{-1} - v, \end{cases}$$

where $\bar{A} = \sqrt{\mu}AD$ and $\bar{Q} = \sqrt{\mu}DQD$ with $D := \text{diag}(d)$ and

$$\delta(xz; \mu) := \delta(v) = \frac{1}{2} \|v^{-1} - v\|.$$

Now observe that

$$d_x^T d_z = d_x^T \bar{Q} d_x \geq 0,$$

since

$$d_x^T d_z = d_x^T (\bar{Q} d_x - \bar{A}^T d_y) = d_x^T \bar{Q} d_x \geq 0,$$

with $\bar{A} d_x = 0$ and \bar{Q} is positive semidefinite.

This last inequality shows that Δx and Δz are not orthogonal directions in contrast with LO case. Thus makes the analysis of the algorithm for CQP more difficult.

Next, we need the following technical lemma that will be used later in the analysis of the algorithm. For its proof the reader can refer to [Lemma C.4 in 8].

Lemma 3.1. *Let (d_x, d_z) be a solution of (6) and if $\delta := \delta(xz; \mu)$ and $\mu > 0$. Then one has*

$$(7) \quad 0 \leq d_x^T d_z \leq 2\delta^2$$

and

$$(8) \quad \|d_x d_z\|_\infty \leq \delta^2, \quad \|d_x d_z\| \leq \sqrt{2}\delta^2.$$

In the next lemmas, we state conditions which ensure the strict feasibility of a full-Newton step.

Lemma 3.2. *Let (x, z) be a strictly feasible primal-dual point. If*

$$e + d_x d_z > 0,$$

then $x_+ = x + \Delta x > 0$ and $z_+ = z + \Delta z > 0$.

Proof. To show that x_+ and z_+ are positive, we introduce a step length $\alpha \in [0, 1]$ and we define

$$x^\alpha = x + \alpha \Delta x, \quad z^\alpha = z + \alpha \Delta z.$$

So $x^0 = x, x^1 = x_+$ and similar notations for z , hence $x^0 z^0 = xz > 0$. We have

$$x^\alpha z^\alpha = (x + \alpha \Delta x)(z + \alpha \Delta z) = xz + \alpha(x\Delta z + z\Delta x) + \alpha^2 \Delta x \Delta z.$$

Now by using (4),and (5) we get

$$x^\alpha z^\alpha = xz + \alpha(\mu e - xz) + \alpha^2 \Delta x \Delta z.$$

We assume that $e + d_x d_z > 0$, we deduce that $\mu e + \Delta x \Delta z > 0$ which is equivalent to $\Delta x \Delta z > -\mu e$. By substitution we obtain

$$\begin{aligned} x^\alpha z^\alpha &> xz + \alpha(\mu e - xz) - \alpha^2 \mu e \\ &= (1 - \alpha)xz + (\alpha - \alpha^2)\mu e \\ &= (1 - \alpha)xz + \alpha(1 - \alpha)\mu e. \end{aligned}$$

Since xz and μe are positive it follows that $x^\alpha z^\alpha > 0$ for $\alpha \in [0, 1]$. Hence, none of the entries of x^α and z^α vanish for $\alpha \in [0, 1]$. Since x^0 and z^0 are positive, this implies that $x^\alpha > 0$ and $z^\alpha > 0$ for $\alpha \in [0, 1]$. Hence, by continuity the vectors x^1 and z^1 must be positive which proves that x_+ and z_+ are positive. This completes the proof. \square

Now for convenience, we may write

$$v_+^2 = \frac{x_+ z_+}{\mu}$$

and it is easy to have

$$v_+^2 = e + d_x d_z.$$

Lemma 3.3. *If $\delta := \delta(x, z; \mu) < 1$. Then $x_+ > 0$ and $z_+ > 0$.*

Proof. We have seen in Lemma 3.2, that $x_+ > 0$ and $z_+ > 0$ are strictly feasible if $e + d_x d_z > 0$. So $e + d_x d_z > 0$ holds if $1 + (d_x d_z)_i > 0$ for all i . In fact we have

$$\begin{aligned} 1 + (d_x d_z)_i &\geq 1 - |(d_x d_z)_i| \text{ for all } i, \\ &\geq 1 - \|d_x d_z\|_\infty \end{aligned}$$

and by the bound in (9) it follows that

$$1 + (d_x d_z)_i \geq 1 - \delta^2.$$

Thus $e + d_x d_z > 0$ if $\delta < 1$. This completes the proof. \square

The next lemma shows the influence of a full-Newton step on the proximity measure.

Lemma 3.4. *If $\delta(xz; \mu) < 1$. Then*

$$\delta_+ := \delta(x_+ z_+, \mu) \leq \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}.$$

Proof. We have

$$\begin{aligned} 4\delta_+^2 &= \|v_+^{-1} - v_+\|^2 \\ &= \|v_+^{-1}(e - v_+^2)\|^2. \end{aligned}$$

But $v_+^2 = e + d_x d_z$ and $v_+^{-1} = \frac{1}{\sqrt{e + d_x d_z}}$, then it follows that

$$4\delta_+^2 = \left\| \frac{d_x d_z}{\sqrt{e + d_x d_z}} \right\|^2 \leq \frac{\|d_x d_z\|^2}{1 - \|d_x d_z\|_\infty}.$$

Now in view of Lemma 3.1, we deduce that $4\delta_+^2 \leq \frac{2\delta^4}{1-\delta^2}$.
This completes the proof. \square

Corollary 3.1. *If $\delta := \delta(xz; \mu) < \frac{1}{\sqrt{2}}$. Then $\delta_+ \leq \delta^2$ which means the quadratic convergence of the proximity measure during a full-Newton step.*

In the next lemma, we discuss the influence on the proximity measure of an update barrier parameter $\mu_+ = (1 - \theta)\mu$ during the Newton process along the central path.

Lemma 3.5. *If $\delta(xz; \mu) < \frac{1}{\sqrt{2}}$ and $\mu_+ = (1 - \theta)\mu$ where $0 < \theta < 1$. Then*

$$\delta^2(x_+ z_+; \mu_+) \leq (1 - \theta)\delta_+^2 + \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{\theta}{2}.$$

In addition if $\delta \leq \frac{1}{\sqrt{2}}$, $\theta = \frac{1}{2\sqrt{n}}$ and $n \geq 2$, then we have:

$$\delta(x_+ z_+; \mu_+) \leq \frac{1}{\sqrt{2}}.$$

Proof. Let $v_+ = \sqrt{\frac{x_+z_+}{\mu}}$ and $\mu_+ = (1 - \theta)\mu$. Then

$$\begin{aligned}
4\delta^2(x_+z_+; \mu_+) &= \left\| \left(\sqrt{\frac{\mu_+}{x_+z_+}} \right) - \left(\sqrt{\frac{x_+z_+}{\mu_+}} \right) \right\|^2 \\
&= \left\| \sqrt{1 - \theta}v_+^{-1} - \frac{1}{\sqrt{1 - \theta}}v_+ \right\|^2 \\
&= \left\| \sqrt{1 - \theta}(v_+^{-1} - v_+) - \frac{\theta}{\sqrt{1 - \theta}}v_+ \right\|^2 \\
&= (1 - \theta) \|v_+^{-1} - v_+\|^2 + \frac{\theta^2}{1 - \theta} \|v_+\|^2 - 2\theta(v_+^{-1} - v_+)^T v_+ \\
&= (1 - \theta) \|v_+^{-1} - v_+\|^2 + \frac{\theta^2}{1 - \theta} \|v_+\|^2 - 2\theta(v_+^{-1})^T v_+ + v_+^T v_+ \\
&= 4(1 - \theta)\delta_+^2 + \frac{\theta^2}{1 - \theta} \|v_+\|^2 - 2\theta n + 2\theta \|v_+\|^2,
\end{aligned}$$

since $(v_+^{-1})^T v_+ = n$ and $v_+^T v_+ = \|v_+\|^2$. Now, recall that $\delta_+^2 \leq \frac{\delta^4}{2(1 - \delta^2)}$. Then

$$4\delta^2(x_+z_+; \mu_+) \leq 4(1 - \theta) \frac{\delta^4}{2(1 - \delta^2)} + \frac{\theta^2}{1 - \theta} \|v_+\|^2 - 2\theta n + 2\theta \|v_+\|^2.$$

Finally, since

$$x_+^T z_+ = \mu n + \mu d_x^T d_z,$$

and if $\delta < \frac{1}{\sqrt{2}}$, it follows by (8) in Lemma 3.1 that

$$\|v_+\|^2 = \frac{1}{\mu} x_+^T z_+ \leq (n + 1),$$

and consequently

$$4\delta^2(x_+z_+; \mu_+) \leq 4(1 - \theta) \frac{\delta^4}{2(1 - \delta^2)} + \frac{\theta^2(n + 1)}{1 - \theta} - 2\theta n + 2\theta(n + 1)$$

and

$$\delta^2(x_+z_+; \mu_+) \leq (1 - \theta)\delta_+^2 + \frac{\theta^2(n + 1)}{4(1 - \theta)} + \frac{\theta}{2}.$$

For the last statement the proof goes as follows. If $\delta < \frac{1}{\sqrt{2}}$, then $\delta_+^2 < \frac{1}{4}$ and this yields the following upper bound for $\delta^2(x_+z_+; \mu_+)$ as:

$$\delta^2(x_+z_+; \mu_+) \leq \frac{(1 - \theta)}{4} + \frac{\theta^2(n + 1)}{4(1 - \theta)} + \frac{\theta}{2}.$$

Now, taking $\theta = \frac{1}{2\sqrt{n}}$ then $\theta^2 = \frac{1}{4n}$, it follows that:

$$\delta^2(x_+z_+; \mu_+) \leq \frac{\frac{1}{4n}(n+1)}{4(1-\theta)} + \frac{\theta}{2} + \frac{(1-\theta)}{4},$$

and since $\frac{n+1}{4n} \leq \frac{3}{8}$ for all $n \geq 2$, then we have:

$$\delta^2(x_+z_+; \mu_+) \leq \frac{3}{32(1-\theta)} + \frac{\theta}{2} + \frac{(1-\theta)}{4}.$$

Now for $n \geq 2$, we have $0 \leq \theta \leq \frac{1}{2\sqrt{2}}$ and since the function

$$f(\theta) = \frac{3}{32(1-\theta)} + \frac{(1-\theta)}{4} + \frac{\theta}{2}$$

is continuous and monotonic increasing on $0 < \theta < \frac{1}{2\sqrt{2}}$, consequently

$$f(\theta) \leq f\left(\frac{1}{2\sqrt{2}}\right) \approx 0.48341 < \frac{1}{2}, \text{ for all } \theta \in \left[0, \frac{1}{2\sqrt{2}}\right].$$

Hence $\delta(x_+z_+; \mu_+) < \frac{1}{\sqrt{2}}$ and the algorithm is well-defined. This completes the proof.

In the next lemma we analyze the effect of a full-Newton step on the duality gap.

Lemma 3.6. *Let $\delta := \delta(xz; \mu) < \frac{1}{\sqrt{2}}$ and $x_+ = x + \Delta x$ and $z_+ = z + \Delta z$. Then the duality gap satisfies*

$$(9) \quad x_+^T z_+ \leq \mu(n+1).$$

Proof. It follows straightforwardly from the proof in Lemma 3.5. \square

In the next lemma we compute a bound for the number of iterations of Algorithm 2.1.

Lemma 3.7. *Let x^{k+1} and z^{k+1} be the $(k+1)$ -th iteration produced by Algorithm 2.1 with $\mu := \mu_k$. Then*

$$(x^{k+1})^T z^{k+1} \leq \epsilon$$

if

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{\mu_0(n+1)}{\epsilon} \right\rceil.$$

Proof. It follows from the bound(10) in Lemma 3.6 that:

$$(x^{k+1})^T z^{k+1} \leq \mu_k(n+1)$$

with

$$\mu_k = (1-\theta)\mu_{k-1} = (1-\theta)^k \mu_0.$$

Then it follows that:

$$(x^{k+1})^T z^{k+1} \leq (1 - \theta)^k \mu_0(n + 1).$$

Thus the inequality $(x^{k+1})^T z^{k+1} \leq \epsilon$ is satisfied if

$$(1 - \theta)^k \mu_0(n + 1) \leq \epsilon.$$

Now taking logarithms of $(1 - \theta)^k \mu_0(n + 1) \leq \epsilon$, we may write

$$k \log(1 - \theta) \leq \log \epsilon - \log \mu_0(n + 1)$$

and using the fact that $\log(1 - \theta) \leq -\theta$, for $0 \leq \theta < 1$ then the above inequality holds if

$$k\theta \geq \log \mu_0(n + 1) - \log \epsilon = \log \frac{(n + 1)\mu_0}{\epsilon}.$$

This completes the proof. \square

For $\theta = \frac{1}{2\sqrt{n}}$, we obtain the following theorem

Theorem 3.1. *Let $\theta = \frac{1}{2\sqrt{n}}$. Then Algorithm 2.1 requires at most*

$$O\left(\sqrt{n} \log \frac{(n + 1)\mu_0}{\epsilon}\right)$$

iterations.

Proof. By replacing $\theta = \frac{1}{2\sqrt{n}}$ in Lemma 3.7, the result holds. \square

4. CONCLUSION AND FUTURE WORKS

In this paper, we have proposed a feasible short-step primal-dual interior point algorithm for solving CQP. The algorithm deserves the best well-known theoretical iteration bound $O(\sqrt{n} \log \frac{(n+1)\mu_0}{\epsilon})$ when the starting point is $x_0 = z_0 = e$. This choice of initial point can be done by the technique of embedding. Future research might extended the algorithm for other optimization problems and its numerical implementation is also an interesting topic.

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