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BAYES-NASH EQUILIBRIUM IN THE PRESENCE OF INFORMATION SOURCES: COMPUTATIONAL ISSUES

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Abstract. This paper discusses computational issues of finding Bayes-Nash equilibrium (BNE) in the presence of information sources, which separate game-specific information from environment-based information. A general algorithm is given.

1. Introduction

Bayesian decision theory is concerned with the question of how a decision maker should choose a particular action from a set of possible choices if the outcome of the choice also depends on some unknown state (from the states of the world). In our approach, the decision maker is modeling the information received by the system (i.e. new information) as an *information source* [3]. A decision problem involves one or several information sources. We assume that each person is able to represent his beliefs, as the likelihood of the different n states of the information source, by a subjective discrete probability distribution [6].

The structure of this paper is as follows. After this introductory part, the next section contains background information and notations used throughout the paper. The third section contains a short description of the classical and proposed solution, as well as the steps of an original algorithm based on the externalization of information sources which computes Bayes-Nash equilibrium for a class of games with incomplete information. Finally, the last section draws some conclusions and outlines future research.

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2. Background material

Consider a *game with incomplete information* (as in [2]), denoted by:
$\Gamma_t = (I, (F_i)_{i\in I}, (\Pi_t^i(f,\theta))_{i\in I}, (\Theta_i)_{i\in I}, \mu_t)$,
where:

- I is the set of players, $|I| = m$,
- F_i is the strategy set for player $i, i = \overline{1,m}$, and $F = F_1 \times F_2 \times \cdots \times F_m$ is the the set of all possible strategy profiles;
- $f = (f_1, f_2, \cdots, f_m) \in F$ is a joint strategy or strategy profile;
- Θ_i is the set of types for the player i, and $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_m$ is the joint type space;
- $\theta = (\theta_1, \theta_2, \cdots, \theta_m) \in \Theta$ is the joint type of all players;
- $\Pi_t^i(f,\theta)$ is the payoff function for player i at the moment t if the strategy f and the type combination θ are chosen. Note that the payoff for the player i may depend not only on its type θ_i, but also on the other players' type, denoted by θ_{-i}.
- μ_t - the probability distribution on the set Θ at the moment t. This is the uncertainty-dominated component of the game, which draws our attention in this paper.

In our exposition, we assume that type sets Θ_i are finite; consequently, Θ is a finite set also. $\mu_t(\theta), \theta \in \Theta$ denotes the probability of chosing type combination θ at the moment t. As in [5], we assume, without loss of generality, that players have incomplete information about their opponents' payoffs but have complete information about the strategies of all other players.

A strategy profile $f^*(\theta) = (f_1^*(\theta_1), f_2^*(\theta_2), \cdots, f_m^*(\theta_m))$ constitutes a *Bayes-Nash equilibrium* (see [2] and [4]) of a game Γ_t with incomplete information if the following inequality:

$$\sum_{\theta_{-i}\in\Theta_{-i}} \Pi_t^i(f_i^*(\theta_i), f_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i})\mu_t(\theta_{-i}|\theta_i) \geq$$
$$\sum_{\theta_{-i}\in\Theta_{-i}} \Pi_t^i(f_i(\theta_i), f_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}) \cdot \mu_t(\theta_{-i}|\theta_i)$$

holds for all possible players $i \in I$ and all types $\theta_i \in \Theta_i$ and all strategies $f_i \in F_i$.

The information source S_{t+1} is a possible probability distribution of states at the moment $t+1$. Γ_{t+1}, the game with incomplete information at the moment $t+1$, based on S_{t+1}, can be defined recursively as follows:

$\Gamma_{t+1} = \Gamma_t(S_{t+1})$, where $\Gamma_{t+1} = (I, (F_i)_{i\in I}, (\Pi_t^i(f,\theta))_{i\in I}, (\Theta_i)_{i\in I}, \mu c_t)$.

The probability distribution μ_t conditioned by the information source S_{t+1}, denoted by μc_t, is the probability distribution μ_t updated by the information source S_{t+1}. The game Γ_{t+1} is the updated game Γ_t based upon S_{t+1}. This information source updates probability distribution μ_t on Θ and thus the equilibrium of Γ_t is modified.

The above notations allow us to define the *Bayes - Nash equilibrium* of the game Γ_{t+1} as a list of decision functions $(f_1^*(.), \cdots, f_m^*(.))_{t+1}$, such that for all possible players $i \in I$ and all types $\theta_i \in \Theta_i$, the inequality:

$$\sum_{\theta_{-i} \in \Theta_{-i}} \Pi_t^i \left(f_i^*(\theta_i), f_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i} \right) \cdot \mu c_t(\theta_{-i}|\theta_i) \geq$$
$$\sum_{\theta_{-i} \in \Theta_{-i}} \Pi_t^i \left(f_i, f_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i} \right) \cdot \mu c_t^i(\theta_{-i}|\theta_i)$$

holds for all strategies $f_i \in F_t$.

The equilibrium of Γ_{t+1} can differ from the equilibrium of Γ_t due to the information in S_{t+1}. For a given player $i \in I$ the updated equilibrium of Γ_{t+1} is:

$f_i^*(.) = \sum\limits_j f_{ij}^*(.) \cdot p^j(t)$,

where $f_{ij}^*(.)$ is the equilibrium of player i for the state s^j of the information source S_{t+1} and $p^j(t)$ is the probability that $S_{t+1} = s^j$.

3. Computing the equilibrium

3.1. Classical approach. According to [7], the computing of equilibrium for a game with incomplete information involves the following steps:

(1) Specify a computational mechanism;
(2) Generate candidate strategies;
(3) Estimate the empirical game;
(4) Solve the empirical game;
(5) Analyze the results.

One of the drawbacks of this approach is that it does not emphasize the dynamics of uncertainty, i.e. the transformation $\mu_t \to \mu_{t+1}$ is hidden. Once the equilibrium is computed at moment t, it is hard to decide when to compute it again (i.e. to decide when the moment $t+1$ is arrived).

The transition $t \to t+1$ is purely formal: the re-computing of equilibria is performed each time it is needed; this is not discussed explicitely in the literature. Our proposal, i.e. the externalization of information sources offers a more precise trigger to re-compute equilibria.

3.2. Externalization of information. The proposed approach models the information environment with the help of n *information sources.* Input data considered are the same as in classical approach: historical data, player data, including payoff functions and strategies, probability distribution $\mu_t = \mu(t)$ and the computed equilibrium at the moment t. Our approach considers the process of updating probability distribution at the $t+1$ moment, $\mu_{t+1} = \mu(t+1)$, as a separate step; this update process uses estimates of information sources for the $t+1$ moment.

The most important thing here, described using a simple example in [8], is the way in which the uncertain component of the game, μ_t is updated (or estimated). The intrinsic uncertainty of this component is due to the fact that information which influences its probability distribution is not entirely known. The solution of the game at the moment $t+1$ is given by the computing of the component μ_{t+1}, which is (seems to be) an update of the distribution μ_t based on the predictions regarding information sources for the moment $t+1$.

This way, the game at the moment $t+1$ is the game at the moment t with the uncertain component μ updated: $\Gamma_{t+1} = \Gamma_t(\mu_t \rightarrow \mu_{t+1})$.

The transition $t \rightarrow t+1$ is purely formal and has the following semantics: when the state of an information source (i.e. the probability distribution of the random variable associated to it) is changed, the equilibrium needs to be re-computed.

Algorithm 1: Computing Bayes-Nash equilibrium in the presence of information sources

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Data: $m \in \mathbf{N}$ number of players; $n \in \mathbf{N}$ number of information sources; $F_1, F_2, \cdots, F_m$ strategies of the players; $\Theta_1, \Theta_2, \cdots, \Theta_m$ types of the players; $(\mu c_0^1, \mu c_0^2, \cdots, \mu c_0^n)$ probability distributions of states of information sources; $(\overline{p_1}, \overline{p_2}, \cdots, \overline{p_m})_{t=0}$ average prices; $\Pi_i, i = \overline{1,m}$ payoff functions; T prognosis (planning) horizon

Result: $Sol = \{(p_1^*, p_2^*, \cdots, p_m^*)_t, t = \overline{1,T}\}$.

begin

    $t \longleftarrow 0$

    while $t < T$ **do**

        { *Step* 1: estimate $\mu c_{t+1}^j, j = \overline{1,n}$ and $(\overline{p_1}, \overline{p_2}, \cdots, \overline{p_m})_{t+1}$}

        { *Step* 2: build the complex information source SC_{t+1}}

        { *Step* 3: update the probability distributions $\mu c_{t+1}^j, j = \overline{1,n}$}

        { *Step* 4: rebuild the payoff functions $\Pi_i, i = \overline{1,m}$}

        { *Step* 5: compute the equilibrium prices $(p_1^*, p_2^*, \cdots, p_m^*)_{t+1}$}

        $t \longleftarrow t+1$

Remarks

(1) Each player produces a single product, slightly different from the products provided by the other players. The strategies of the players are represented by the equilibrium prices $(p_1^*, p_2^*, \cdots, p_m^*)_t$ for their products at each moment t in the planning horizon, $t = \overline{1, T}$.
(2) In *Step* 3, probability distributions μc_{t+1}^j are conditioned by the state of the complex information source SC_{t+1}.
(3) In *Step* 4, if payoff functions are given in analytical form, they need to be rebuilt, as shown in [8].
(4) The estimation of the equilibrium in *Step* 5 depends on the form of payoff functions. If they are continuously differentiable with respect to the prices and an analytical solution can be found, as shown in [8], then the solution is straightforward. If payoff functions are given in table form, or an analytical solution is not known, the general approach is to reduce the game Γ_{t+1} to a game with complete information (as discussed in [1]) and to compute its Nash equilibrium using known algorithms. This single step is subject to recent research for particular games.
(5) Another key issue in implementing this algorithm is the representation of the components of the game. Simple examples involving two players and several information sources are easy to manage, but the space required is growing at least polynomial in the number of players and information sources considered.

4. Conclusions and further work

The essential difference between the classical approach and the one proposed in this paper is given by the separation of the information external to the game from the game-specific information. This separation follows the *separation of responsibilities* principle. This way, both external and internal elements of the game are easier to model and understand.

The classical approach does not make any distinction between these two categories of information; more precisely, the influence of external information on the uncertainty that dominates the game is not taken into account or quantified. By splitting the game information into external and internal, the former being modeled by information sources, the influence of external environment on the variation of the solution is better captured and quantified. This provides a better evaluation of the contribution of individual factors to the predicted equilibrium.

Another advantage of this separation is that it allows a better, easier calibration of the model, by comparing the computed equilibrium with real solution, taken from historical data.

Further work will include the detailed study of concrete problems involving games with incomplete information, and comparing the results obtained using classical and proposed approaches. The general algorithm presented here will be implemented in several game-specific applications.

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