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AN EVOLUTIONARY APPROACH OF DETECTING SOME REFINEMENTS OF THE NASH EQUILIBRIUM

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Abstract. In non-cooperative game theory the most important solution concept is the Nash equilibrium. Many refinements of this one are introduced in order to solve the selection problem associated to the games having several Nash equilibria. Numerical experiments are proposed to calculate the distance between the Pareto front and different type of equilibria. A generative relation and an evolutionary technique for detection different Nash equilibrium refinements are used. The experiments show that these equilibria concepts can be useful in multi-objective optimization problems.

1. Introduction

A major application of Game Theory is the equilibrium detection. In general an equilibrium can be described as a state, from that no player wants to deviate. The most known equilibrium concept in non-cooperative Game Theory is the Nash equilibrium [8]. Intuitively, a strategy profile is a Nash equilibrium if there is no player who can change his/her strategy in order to improve his/her payoff. For games having more Nash equilibria can appear a selection problem. Agents can not decide which strategy to play, therefore can appear bad decisions. Several refinements of the Nash equilibrium have been developed: Aumann (strong Nash) equilibrium [1], coalition proof Nash equilibrium [2].

Our goal is to compare the detected different equilibria types with the Pareto front of the experiments.

2. Game theoretic prerequisites

A finite strategic non-cooperative game, $G = (N, S_i, u_i, i = 1, ..., n)$, can be described as a system, where:

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- N represents a set of players, and n is the number of players;
- for each player $i \in N$, S_i is the set of available actions, $S = S_1 \times S_2 \times ... \times S_n$ is the set of all possible situations of the game and $s \in S$ is a strategy (or strategy profile) of the game;
- for each player $i \in N$, $u_i : S \to R$ represents the payoff function (utility) of the player i.

In the following we present shortly the different equilibria types.

2.1. **Pareto efficiency.** The solution is Pareto-efficient if there is no possibility of improving the payoff of one agent, without making that of another agent worse.

2.2. **Nash equilibrium.** As we mentioned Nash equilibrium is a strategy profile from then no player can deviate in order to increase her/his payoff.

Formally:

Definition 1. *A strategy profile $s^* \in S$ is a Nash equilibrium if the inequality holds:*

$$u_i(s_{ij}, s^*_{-i}) \leq u_i(s^*), \forall i = 1, ..., n, \forall s_{ij} \in S_i,$$

where (s_{ij}, s^*_{-i}) denotes the strategy profile obtained from s^* by replacing the strategy of player i with s_{ij}.

2.3. **Aumann (strong Nash) equilibrium.** The Aumann equilibrium is a game strategy for which no coalition of players has a joint deviation that improve the payoff of each member of the coalition.

In order to give a formal definition,let (s_I, s^*_{-I}) denotes the strategy profile in which $i \in I$ chooses the individual strategy s_i, and each $j \in N - I$ chooses s^*_j.

Definition 2. *The strategy s^* is an Aumann equilibrium if for each coalition $I \subseteq N, I \neq \phi$ the inequality*

$$u_i(s_I, s^*_{-I}) \leq u_i(s^*), \forall i \in I,$$

holds.

2.4. **Coalition proof Nash equilibrium.** Bernheim [2] introduced the coalition proof Nash equilibrium. A coalition-proof equilibrium is a correlated strategy from which no coalition has an improving and self-enforcing deviation.

Definition 3. *Let $s^* \in S$ and let P be the set of the subsets of $\{1,2,...n\}$. An internally consistent improvement (ICI) of P upon s^* is defined by induction on card(P) [6]:*

- *if* $card(P) = 1$, *then* $P = \{i\}$, *then* s_i *is an* ICI *upon* s^*, *if*

$$u_i(s_i, s^*_{N-i}) > u_i(s^*);$$

- *if* $card(P) > 1$, *then* $s^P \in S^P$ *is an* ICI *of* P *upon* s^*
  (i) s^P *is an improvement of* P *upon* s^*;
  and
  (ii) if $T \subset P$ *and* $card(T) < card(S)$ *then* T *has no* ICI *upon* (s^P, s^*_{N-S}).

Definition 4. *A strategy profile* $s \in S$ *is a coalition proof Nash equilibrium, if no* P *subcoalition has an* ICI *upon* s^*.

3. Evolutionary equilibria detection

In order to obtain the above mentioned equilibria types of a non-cooperative game we define generative relations.

Several generative relations are introduced, for Nash equilibrium [7], for Aumann equilibrium [4], for modified strong Nash and coalition proof Nash equilibrium [5]. Generative relations may be used for ranking-based fitness assignment in an evolutionary technique for equilibria detection.

3.1. Generative relation for different equilibria types.

Consider two strategy profiles s and s^* from S.

We may express the generative relation generative relation (s, s^*) as the number of players or coalition of players for which some players or coalitions of players change from the initial strategy.

Generative relations for the certain equilibria are the following:

- Nash equilibrium

$$k(s^*, s) = card\{i \in \{1, ..., n\} | u_i(s_i, s^*_{-i}) \geq u_i(s^*), s_i \neq s^*_i\}.$$

- Aumann equilibrium

$$a(s, s^*) = card[i \in I, \phi \neq I \subseteq N, u_i(s^*_I, s_{-I}) \geq u_i(s), s^*_i \neq s_{-i}],$$

- coalition proof Nash equilibrium

$$cn(s^*, s) = card[i \in I, \phi \neq I \subseteq N, u_i(s_I, s^*_{-I}) \geq u_i(s^*), s_i \neq s^*_{-i}]$$

$$+card[t \in T, T \neq \phi, T \subset I, \phi \neq I \subseteq N, u_t(z_t, s_{I-T}, s^*_{N-I}) \geq u_t(s_I, s^*_{N-I}),$$

$$s_I \neq s^*_I, z_t \in S_T],$$

Definition 5. *Let $s, s^* \in S$. We say the strategy s is better than strategy s^* with respect to the certain equilibrium, and we write $s \prec_{EQ} s^*$, if and only if the inequality*

generative relation(s, s^*) $<$*generative relation*(s^*, s),

holds.

Definition 6. *The strategy profile $s^* \in S$ is a certain non-dominated strategy, if and only if there is no strategy $s \in S, s \neq s^*$ such that s dominates s^* with respect to $\prec_{EQ}$ i.e.*

$$s \prec_{EQ} s^*.$$

We may consider the relation $\prec_{EQ}$ as a generative relation of the certain equilibrium. The set of the certain equilibria equals the set of the nondominant strategies with respect to the relation $\prec_{EQ}$. We may consider the set of non-dominated strategies as a subset of the certain equilibrium of the game.

3.2. **Evolutionary equilibrium detection method.** A population of strategies is evolved using the dominance concept based on the generative relation.

The individuals in the Pareto front are represented as an n-dimensional vector representing a strategy profile $s \in S$.

An initial population is generated randomly. A subsequent application of the such operators (like the simulated binary crossover (SBX) and real polynomial mutation [3]) is guided by a specific selection operator induced by the generative relation.

At iteration t the strategy population may be regarded as the current equilibrium approximation (Nash, Aumann or coalition proof Nash equilibrium). The successive populations produce new approximations of the equilibrium front.

4. Numerical experiments

We would like to examine the position of the Pareto front to the Nash equilibrium and to its refinements. In each experiment the population size is 200 the maximal number of generation is 50.

4.1. **Experiment 1.** Let us consider game G_1, having the following payoff functions [4]:

$$u_i(x_1, x_2) = x_i[10 - sin(x_1 + x_2)], x_i \in [0, 10], i = 1, 2.$$

We have detected all of the Aumann and coalition proof Nash equilibria on the Pareto front, and some of the Nash equilibria lies on the Pareto front and some of it under the Nash equilibria.

	Nash eq.	Aumann eq.	coalition proof Nash eq.
G_1	0	0	0
G_2	0	0	0

TABLE 1. Minimum distance from the Pareto front in the case of the Nash, Aumann and coalition proof Nash equilibrium in the best population

	Nash eq.	Aumann eq.	coalition proof Nash eq.
G_1	3.0181	0.3769	0.35759
G_2	49543	0	0

TABLE 2. Maximum distance from the Pareto front in the case of the Nash, Aumann and coalition proof Nash equilibrium in the best population

4.2. **Experiment 2.** Let us consider the three person game G_2 with the following payoff functions:

$$u_i = e^{x_i}(a - sin(\sum_{i=1,3} x_i^2)), x_i \in [0, 10], i = 1, 2, 3, a = 1;$$

We have detected only one coalition proof Nash and Aumann equilibrium the strategy $(10, 10, 10)$ with the corresponding payoff $(44047.55, 44047.55, 44047.55)$.

4.3. **Numerical results.** Table 1 and 2 presents the distance between the Pareto front and the minimum and maximum values of the different equilibria payoffs in the final population.

The numerical experiments show that the certain equilibrium detection can be a good tool in optimization problems, as well. The Pareto front contains an infinite number of points, the refinements of the Nash equilibrium (Aumann and coalition proof Nash equilibria) reduce the set of the solutions.

5. CONCLUSIONS

Generative relations are used for evolutionary equilibrium detection. The detected different type of equilibria can be solutions in multi-objective optimization problems.

Numerical experiments show that detected equilibria can be better solutions in some cases than Pareto front detection. In the most cases Pareto front contains an infinite number of values, the different refinements of the Nash equilibria gives less solutions.

Calculating the minimum and maximum distance from the Pareto front to the different equilibria types in the best population we can conclude that some of the refinements of the Nash equilibrium lie on the Pareto front. In the presented games the number of the Nash refinement solutions is small, therefore these solution concepts can be viewed as a new optimization tool. Further work will include experiments with more players, and other equilibrium concepts.

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