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NONDOMINATION IN LARGE GAMES: BERGE-ZHUKOVSKII EQUILIBRIUM

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Abstract. Generative relations, a class of binary relations on the game strategies, can characterize game equilibria. The set of non dominated strategies with respect to the generative relation describes the game equilibrium. Evolutionary techniques based on nondomination may detect game equilibria. Some properties of a generative relation used to detect Berge-Zhukovskii equilibrium are investigated. Numerical results on a Cournot model illustrate the proposed techniques.

1. Introduction

Among the most popular solutions in game theory are equilibria [4] such as Nash or Aumann equilibrium. Each of them cope with different situations and game conditions regarding players rationality however they give sometime unrealistic results predicting how real players choose their actions.

In Berge-Zhukovskii equilibrium agents are allowed to play in a cooperative way. By allowing them to form coalitions the equilibrium describes a kind of reciprocation altruism and it represents a robust solution concept in Game Theory, more close to people behaviour than Nash equilibrium.

Large games (with a great number of players) are of great interest but the computational costs for finding good approximations of equilibria are extremely high.

A fitness concept based on non-domination has been proposed for strategic games in normal form for pure strategies. Similar with the Pareto dominance from the evolutionary multi objective algorithms [1] a domination relation between two members of the population has been defined. This relation (named generative relation) allows a comparison between two individuals.

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Pareto-dominance concept is known to be inefficient for many-objectives optimization problems. Algorithms that use Pareto-dominance are inefficient for more than 3, 4 objectives [2].

The difficulties in computing Berge equilibrium for large games are presented and studied here in comparison with Pareto-dominance for many-objectives optimization.

2. Prerequisites

Some basic notions from Game Theory are considered (see, for instance, [4]).

2.1. **Strategic games. Definition.** A finite *strategic game* is defined as a system by $G = (N, S, U)$ where:

- $N = \{1, ..., n\}$, represents the set of players, n is the number of players;
- for each player $i \in N$, S_i represents the set of actions available to her;
- $S = S_1 \times S_2 \times ... \times S_n$ is the set of all possible situations of the game;
- $(s_1, s_2, ..., s_n) \in S$ is a strategy profile.
- for each player $i \in N$, $u_i : S \to \mathbf{R}$ represents the payoff function.

$$U = \{u_1, ..., u_n\}.$$

Remark. In a strategic game the set of all possible strategy profiles represents the search space.

2.2. **Berge-Zhukovskii equilibrium.** Berge-Zhukovskii equilibrium [6] can be viewed as a solution for games that do not have a Nash equilibrium, or for games which have more than one Nash equilibrium.

The strategy s^* is a Berge equilibrium in the sense of Zhukovskii, (or Berge-Zhukovskii equilibrium) if at least one of the players of the coalition $N - \{i\}$ deviates from her equilibrium strategy, the payoff of the player i in the resulting strategy profile would be at most equal to her payoff $u_i(s^*)$ in the equilibrium strategy.

Formally we may write:

Definition. A strategy profile $s^* \in S$ is a Berge-Zhukovskii equilibrium if the inequality

$$u_i(s^*) \geq u_i(s_i^*, s_{N-i})$$

holds for each player $i = 1, ..., n$, and $s_{N-i} \in S_{N-i}$.

A player that chooses a strategy from Berge-Zhukovskii equilibrium obtains a maximum payoff when the other players also choose strategies from Berge-Zhukovskii equilibrium.

3. Generative relations

Two generative relations are presented in this section, relations used to guide the search towards the equilibria: the Pareto front and the Berge-Zhukovskii-equilibrium respectively. With respect to these relations, two strategy profiles can be indifferent one to another, or one of them dominated by the other.

3.1. Pareto domination.

Definition. A strategy s' Pareto-dominates the strategy s'' if and only if each player has a better payoff for s' than for s''. We write $s' \leq_P s''$ or $(s', s'') \in P_d$.

Formally s' Pareto dominates s'' if and only if we have $u_i(s') \geq u_i(s'')$ $\forall i \in \{1, ..., n\}$ and there $\exists j \in \{1, ..., n\} : u_j(s') > u_j(s'')$.

A strategy s'' is called Pareto-non dominated if $\nexists s' \in S : (s', s'') \in P_d$.

A Pareto-non dominated strategy is also called Pareto optimal or Pareto efficient.

In a similar manner to the Pareto domination relation, two strategy profiles may either dominate each other or they may be indifferent to each other.

3.2. Berge-Zhukovskii equilibrium.

A generative relation for Berge-Zhukovskii equilibrium is presented in this section. The relation is constructed similar to Nash-ascendency relation introduced in [3].

Consider two strategy profiles x and y from S. Denote by $b(x, y)$ the number of players who lose by keeping the initial strategy x, while the other players are allowed to play the corresponding strategies from y.

Consider

$$b(x, y) = card\{i \in N, u_i(x) < u_i(x_i, y_{N-i})\}.$$

Definition. Let $x, y \in S$. We say the strategy x is better than strategy y with respect to Berge-Zhukovskii equilibrium, and we write $x \prec_{BZ} y$, if and only if the inequality

$$b(x, y) < b(y, x),$$

holds.

Definition. The strategy profile $y \in S$ is a Berge-Zhukovskii non-dominated strategy, if and only if there is no strategy $x \in S, x \neq y$ such that x dominates y with respect to $\prec_{BZ}$ i.e. $x \prec_{BZ} y$.

Denote by NBZ the set of all non-dominated strategies with respect to the relation $\prec_{BZ}$. This set equals the set of Berge-Zhukovskii equilibria, if the Berge-Zhukovskii equilibrium exists for that game.

4. Numerical experiments

In order to analyse the domination in an arbitrary population for Nash-ascendency relation we consider a Cournot oligopoly. The generative relations for Pareto and Berge-Zhukovskii equilibria are analysed and compare using the coefficient of relative dominance [5].

Consider P a set of m strategy profiles, $R \subset S_1 \times S_2 \times ... \times S_n$.

In order to compare the relations in the population P we consider the coefficient of relative dominance

$$K_{rd} = \frac{D}{T}$$

where D denotes the number of pairs from P in which one individual dominates the other with respect to the relation, and T the total number of pairs of individuals from P.

The coefficient of relative dominance is a good indicator of the potential of a generative relation.

4.1. Cournot model of oligopoly.

A single good is produced by n firms. The cost to firm i of producing q_i units of the good is $C_i(q_i)$, where C_i is an increasing function (more output is more costly to produce). All the output is sold at a single price, determined by the demand for the good and the firms total output. If the firms total output is Q, than the market price is $P(Q)$.

If the output of each firm is q_i, then the price is $P(q_1 + q_2 + ... + q_n)$ and the firm i' revenue is $q_i P(q_1 + q_2 + ... + q_n)$. The payoff function for the player i is:

$$\begin{aligned} \pi_i(q_1, ..., q_n) &= q_i P(Q) - C_i(q_i) \\ &= q_i \left[a - (q_1 + ... + q_n) - c\right]. \end{aligned}$$

Each firm cost function is $C_i(q_i) = c * q_i$ for all q_i. $P(Q) = a - Q$ if $Q \leq a$ and 0 otherwise. We consider in the following experiments that $a = 24$ and $c = 9$.

4.2. Experimental set up.

A population P of 50 individuals is randomly generated. Each member of this population $s \in P$ represents a strategy profile $s = (s_1, s_2, ..., s_n) \in S_1 \times S_2 \times ... \times S_n$ where n is the number of players and $S_i \in [0, 10]$.

We will count: the number of pairs where an individual dominates the other one, and the number of pairs of individuals that are indifferent to each other. We will also compute K_{rd}, the coefficient of relative dominance for both relations. The analysis will be made for different numbers of players, from 2 to 30. Presented results are averages after 30 runs of the algorithm.

Table 1. Results for the Pareto domination and for the generative relation for Berge-Zhukovskii equilibrium in the Cournot game

Number of players	No. of pairs					
	Pareto domination			Berge-Zhukovskii domination		
	dominated	indifferent	K_{rd}	dominated	indifferent	K_{rd}
2	628	597	0.51	684	541	0.55
5	617	608	0.50	870	355	0.71
10	59	1166	0.04	1008	217	0.82
20	0	1225	0.00	1216	9	0.99
30	0	1225	0.00	1225	0	1.00

Pareto Dominance. If we consider the above experimental set-up for the game of Cournot type we obtain the results similar with those in current literature. As the number of players increases, the chances that one individual from a pair dominates the other get extremely low (Table 1).
Berge-Zhukovskii equilibrium. For the Berge-Zhukovskii equilibrium generative relation, as the number of players increases so does K_{rd} and the number of indifferent individuals with respect to the ascendancy relation tends to zero (Table 1).

5. Conclusion

Some properties of the generative relation for Berge-Zhukovskii equilibrium in large game are presented. As the number of players increases, the number of strategy profiles indifferent to each other with respect to the generative relation decreases, unlike the case of Pareto dominance.

The generative relation for Berge-Zhukovskii equilibrium would become problematic because the lack of indifferent individuals, unlike the Pareto dominance relation that becomes useless in many-objective optimization due to too many indifferent individuals. In both cases - for many objectives/players - both relations fail to indicate efficient solutions.

The study of these properties may be useful in improving the results of evolutionary search operators designed for solving large games.

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