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MODIFIED STRONG AND COALITION PROOF NASH EQUILIBRIA. AN EVOLUTIONARY APPROACH

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ABSTRACT. In non-cooperative games one of the most important solution concept is the Nash equilibrium based on the idea of stability against unilateral deviations. In games having more Nash equilibria a selection problem can appear. The modified strong Nash equilibrium and the coalition proof Nash equilibrium are important refinements of the Nash equilibrium that can solve the selection problem. A generative relation for the modified strong Nash and for the coalition proof Nash equilibrium based on nondomination is proposed. Some examples illustrate the effectiveness of the proposed method.

1. Introduction

Game Theory represents a basis for neo-classical microeconomic theory and it is an important research field [11].

A finite strategic game is defined a system $G = ((N, S_i, u_i), i = 1, n)$, where:

- N represents a set of players, and n is the number of players;
- for each player $i \in N$, S_i is the set of available actions,

$$S = S_1 \times S_2 \times ... \times S_n$$

  is the set of all possible situations of the game and $s \in S$ is a strategy (or strategy profile) of the game;
- for each player $i \in N$, $u_i : S \to R$ represents the payoff function (utility) of the player i.

The Nash equilibrium [9] is one of the most important solving concepts in non-cooperative game theory. Playing in Nash sense means that no player has a better chance to improve her payoff while others keep theirs unchanged.

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Definition 1. *A strategy profile $s^* \in S$ is a Nash equilibrium if the inequality holds:*

$$u_i(s_{ij}, s^*_{-i}) \leq u_i(s^*), \forall i = 1, n, \forall s_{ij} \in S_i,$$

where (s_{ij}, s^*_{-i}) denotes the strategy profile obtained from s^* by replacing the strategy of player i with s_{ij}.

The problem of detecting the Nash equilibrium is an important computational task. In [8] Nash equilibrium is characterized by a generative relation.

A selection problem can appear in games having more Nash equilibria. Several refinements have been introduced to solve this selection problem. One of this is the Aumann equilibrium [1].

This paper is concerned with on two refinements of the Nash equilibrium: the modified strong Nash equilibrium and the coalition proof Nash equilibrium.

2. Nash equilibrium refinements

Two important Nash equilibrium refinements are presented in this section: the modified strong Nash equilibrium and the coalition proof Nash equilibrium.

2.1. Modified strong Nash equilibrium.

The modified strong Nash equilibrium is introduced by Ray [12] and Greenberg [6].

Let us consider a finite strategic game $G = ((N, S_i, u_i), i = 1, n)$, and the following notations: $S_I = \prod_{i \in I} S_i$ and $x_I = (x_i)_{i \in I}$.

The following definitions are necessary to introduce the modified strong Nash equilibrium:

Definition 2. *For $I \in 2^N - \{\emptyset\}$, $x \in S_N, y_I \in S_I$ we say that y_I is blocked by $T \subset I$ given x if there exists a vector $z_T \in S_T$ such that:*

$$u_T(z_T, y_{I-T}, x_{N-T}) \geq u_T(y_T, x_{N-T}).$$

Definition 3. *I is credible given x if there is a $y_I \in S_I, y_I \neq x_I$, that is not blocked by any credible $T \subset I$ given x.*

Definition 4. *A strategy profile $x \in S_N$ is a modified strong Nash equilibrium if it is not blocked by any credible coalition (given x).*

Example 1. *Game G_1. Let us consider the two person game, for that the payoffs are represented in Table 1. The game has a pure Nash equilibria (B, B), but this not an Aumann equilibrium. However this is a modified strong equilibrium, because it can be blocked by any credible coalition.*

TABLE 1. The payoff functions of the two players for game G_1

		Player2	
		A	*B*
Player1	*A*	*(5,5)*	*(3,6)*
	B	*(6,3)*	*(4,4)*

2.2. **Coalition proof Nash equilibrium.** Bernheim [2] introduced the coalition proof Nash equilibrium. A coalition-proof equilibrium is a correlated strategy from which no coalition has an improving and self-enforcing deviation.

Definition 5. *Let $s^* \in S$ and let P the set of the subsets of N. An internally consistent improvement of P upon s^* is defined by induction on $card(P)$* [7]*:*

- *if $card(P) = 1$, then $P = \{i\}$, then s_i is an ICI upon s^*, if*
$$u_i(s_i, s^*_{N-i}) > u_i(s^*);$$
- *if $card(P) > 1$, then $s^P \in S^P$ is an ICI of P upon s^**
  (i) s^P is an improvement of P upon s^*;
  and
  (ii) if $T \subset P$ and $card(T) < card(S)$ then T has no ICI upon (s^P, s^*_{N-S}).

Definition 6. *A strategy profile $s \in S$ is a coalition proof Nash equilibrium, if no P subcoalition has an ICI upon s^*.*

Let us denote by CNE the coalition proof Nash equilibrium.

Remark 1. *The coalition proof Nash equilibrium is a subset of the Nash equilibrium:*
$$CNE \subseteq NE.$$

3. Generative relations

Generative relations for modified strong Nash and coalition proof Nash equilibria are introduced.

3.1. **Generative relation for modified strong Nash equilibrium.** Consider two strategy profiles x and y from S. Denote by $ms(x, y)$ the number of players in coalition $T, T \subset I,\ I \subseteq N$ benefiting from switching between strategies:

$$ms(x, y) = card[t \in T, T \neq \phi, T \subset I, \phi \neq I \subseteq N,$$
$$u_t(z_t, y_{I-T}, x_{N-I}) \geq u_t(y_I, x_{N-I}), y_I \neq x_I, z_t \in S_T],$$

where $card[M]$ denotes the cardinality of the multiset M (an element i can appear several times in M and each occurrence is counted in $card[M]$).

Definition 7. *Let $x, y \in S$. We say the strategy x is better than strategy y with respect to modified strong Nash equilibrium, and we write $x \prec_{MS} y$, if and only if the inequality*

$$ms(x, y) < ms(y, x),$$

holds.

Definition 8. *The strategy profile $y \in S$ is a modified strong Nash non-dominated (NMSN) strategy, if and only if there is no strategy $x \in S, x \neq y$ such that x dominates y, i.e.*

$$x \prec_{MS} y.$$

We consider relation $\prec_{MS}$ as the generative relation of the modified strong Nash equilibrium. The nondominant strategies with respect to the relation $\prec_{MS}$ can be a suitable representation of the modified strong Nash equilibrium.

3.2. **Generative relation for coalition proof Nash equilibrium.** Consider two strategy profiles x and y from S.

We may define the quality $cn(x, y)$ as:

$$cn(s, s^*) = card[i \in I, \phi \neq I \subseteq N, u_i(y^I, x^{*-I}) \geq u_i(x^*), y^i \neq x^{*-i}]$$
$$+card[t \in T, T \neq \phi, T \subset I, \phi \neq I \subseteq N, u_t(z_t, y_{I-T}, x_{N-I}) \geq u_t(y_I, x_{N-I}),$$
$$y_I \neq x_I, z_t \in S_T],$$

where $card[M]$ denotes the cardinality of the multiset M.

Definition 9. *Let $x, y \in S$. We say the strategy x is better than strategy y with respect to coalition proof Nash equilibrium, and we write $x \prec_{CN} y$, if and only if the inequality*

$$cn(x, y) < cn(y, x),$$

holds.

Definition 10. *The strategy profile $y \in S$ is a coalition proof Nash non-dominated strategy, if and only if there is no strategy $x \in S, x \neq y$ such that x dominates y with respect to $\prec_{CN}$ i.e.*

$$x \prec_{CN} y.$$

We may consider relation $\prec_{CN}$ as a candidate for generative relation of the coalition proof Nash equilibrium.

4. Evolutionary equilibria detection

Generative relations may be used by evolutionary techniques for equilibria detection.

A population of game strategies is evolved. Every individual is encoded as a n-dimensional vector representing a strategy $s \in S$.

An initial strategy population is randomly generated. Population at iteration t may be viewed as the set of current equilibrium approximation.

Simulated binary crossover (SBX) [5] and real polynomial mutation [4] operators are used. The generative relation is used for rank-based fitness assignment.

The evolutionary technique is called the Relational Evolutionary Equilibria Detection (REED), which can be described as follows:

REED method

Step1. Set $t = 0$;
Step2. Randomly initialize a population $P(0)$ of strategies;
Step3. Binary tournament selection and recombination using the simulated binary crossover (SBX) operator for $P(t) \to Q$;
Step4. Mutation on Q using real polynomial mutation $\to P$;
Step5. Compute the rank of each population member in $P(t) \cup P$ with respect to the generative relation. Order by rank $(P(t) \cup P)$;
Step6. Rank based selection for survival $\to$ P (t + 1);
Step7. Repeat steps Step3 - Step6 until the maximum generation number is reached.

5. Numerical experiments

The generative relations are used for the rank based fitness assignment. The population size is 300 and the number of generation is 150. The used parameter setting is described in [4].

The experiments have been conducted for ten runs with different random seed generators.

In order to illustrate the proposed technique some discrete and continuous games are presented.

5.1. **Experiment 1.** Let us consider the following three person game [3], denoted by G_2, where the payoffs are represented in Table 5.1. The first player has three strategies, and her payoff is the first value from the triplet. The second player has three strategies, as well, her payoff is the second value from the triplet. The third player has two strategies her payoff is the third value.

TABLE 2. The payoff values of the three players in the game G_2

			Player2		
			A	B	C
Player3 A	Player1	A	(-2,-2,-10)	(-10,-10,-10)	(-10,-10,-10)
		B	(-10,-10,-10)	(1,1,-5)	(-10,2,-10)
		C	(-10,-10,-10)	(2,-10,-10)	(0,0,10)
			Player2		
			A	B	C
Player3 B	Player1	A	(-1,-1,5)	(-5,-5,0)	(-10,-10,-10)
		B	(-5,-5, 0)	(-2,-2,-10)	(-10,-10,-10)
		C	(-10,-10,-10)	(-10,-10,-10)	(-15,-15,-15)

The game has two pure Nash equilibria (C, C, A), the first player plays C, the second C, and the third plays A. The other Nash equilibrium is (A, A, B). The game has only one modified strong Nash equilibrium (A, A, B), and only one coalition proof Nash equilibrium: (C, C, A).

The algorithm detected correctly all these different types of equilibria.

5.2. **Experiment 2.** Let us consider the game G_3 [10], having the following payoff functions:

$$u_1(x_1, x_2) = -x_1^2 - x_1 + x_2,$$
$$u_2(x_1, x_2) = 2x_1^2 + 3x_1 - x_2^2 - 3x_2, x_1, x_2 \in [0, 1].$$

The corresponding payoffs for the Nash equilibrium, the modified strong Nash equilibrium and the Pareto front are depicted in Figure 1. The Nash equilibrium and the modified strong Nash equilibrium are the same, (0,0) and the corresponding payoff is (0,0).

5.3. **Experiment 3.** Let us consider the three players game G_4, having the following payoff functions:

$$u_1(x, y, z) = x(10 - sin(x^2 + y^2 + z^2)),$$
$$u_2(x, y, z) = y(10 - sin(x^2 + y^2 + z^2)),$$
$$u_3(x, y, z) = z(10 - sin(x^2 + y^2 + z^2)),$$
$$x, y, z, \in [0, 10].$$

This game has more Nash equilibria, and only one modified strong and coalition proof Nash equilibrium, which is the strategy pair $(10, 10, 10)$, and the corresponding payoff $(110, 110, 110)$. The three equilibria types are depicted on Figure 2.

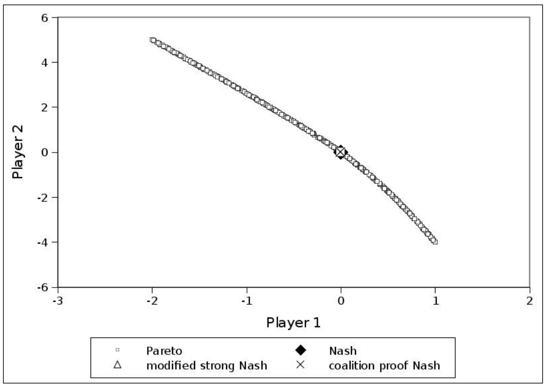

FIGURE 1. Detected payoffs for Pareto front, Nash equilibrium and modified strong Nash equilibrium for the game G_3

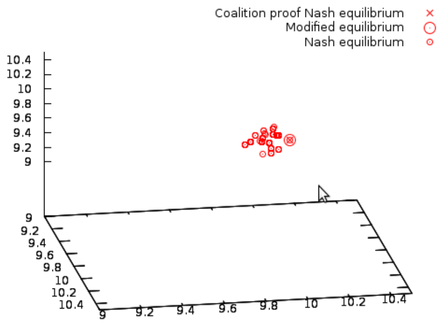

FIGURE 2. Detected strategies for Nash equilibrium, modified strong Nash equilibrium and coalition proof Nash equilibrium for Game G_4

6. Conclusions

The modified strong Nash and the coalition proof Nash equilibrium are refinements of the well-studied Nash equilibrium. Generative relations for modified strong Nash equilibria and of coalition proof Nash equilibrium are proposed.

An evolutionary approach is presented for detecting the modified strong Nash and the coalition proof Nash equilibria. Some discrete and continuous games are considered for numerical experiments. The experiments illustrate the effectiveness of the proposed method. A further step can be simulation of games with more players.

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References

[1] Aumann, R.: *Acceptable Points in General Cooperative n Person Games* , Contributions to the Theory of Games, Vol IV, Annals of Mathematics Studies, 40. 287-324, 1959.
[2] Bernheim B.D,. Peleg B., Whinston M.D.: *Coalition-proof equilibria. I. Concepts.*, J Econ Theory 42: 112, 1987.
[3] Borm, P., Otten, G-J., Peters, H., Larbani, M.: *Core Implementation in Modified Strong and Coalition Proof Equilibria*, Cahiers du Centre d'Etudes de Recherche Operationnelle, 1992, v.34,187-197.
[4] Deb, K., Agrawal, S., Pratab, A., Meyarivan, T.: *A Fast Elitist Non-Dominated Sorting Genetic Algorithm for Multi-Objective Optimization: NSGA-II*, Marc Schoenauer, Kalyanmoy Deb, Günter Rudolph, Xin Yao, Evelyne Lutton, Juan Julian Merelo, and Hans-Paul Schwefel, editors, Proceedings of the Parallel Problem Solving from Nature VI Conference, Paris, France, 2000.
[5] Deb, K., Beyer, H.: *Self-adaptive genetic algorithms with simulated binary crossover*, Complex Systems, 1995, vol.9, 431-454.
[6] Greenberg, J.: *The core and the solution as abstract stable sets*, mimeo, University of Haiffa.
[7] Keiding, H., Peleg, B.: *Representation of effectivity functions in coalition proof Nash equilibrium: A complete characterization*, Soc Choise Welfare (2002) 19, 241-263.
[8] Lung, R. I., Dumitrescu, D.: *Computing Nash Equilibria by Means of Evolutionary Computation*, Int. J. of Computers, Communications & Control, 2008, 364-368.
[9] Nash, J. F.: *Non-cooperative games*, Annals of Mathematics, 54:286-295, 1951.
[10] Nessiah, R., Tazdait, T.: *Strong Berge and Pareto Equilibrium Existence for a Noncooperative Game*, Working paper.
[11] Osborne, M.: *An Introduction to Game Theory*, Oxford University Press, New York, 2004.
[12] Ray, D.: *Credible coalitions and the core* International Journal of Game Theory 18, 185-187.

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