STUDIA UNIV. BABEŞ–BOLYAI, INFORMATICA, Volume **LV**, Number 4, 2010

GLOBAL SEARCH AND LOCAL ASCENT FOR LARGE COURNOT GAMES

MARIA SĂRĂŞAN(1), TUDOR DAN MIHOC(1), RODICA IOANA LUNG(2), AND D. DUMITRESCU(1)

Abstract. Equilibria detection in large games is a fundamental problem in computational game theory. A memetic algorithm, Global Search and Local Ascent (GSLA), is proposed. GSLA's performance is evaluated by means of numerical experiments within the framework of a Cournot game involving up to 100 players and by comparison with an evolutionary multiobjective optimization algorithm adapted for Nash equilibria detection.

1. Introduction

Mathematical games share a lot of similarities with Multi-Objective Optimization Problems (MOOPs). A non-cooperative game consists of a set of players, a set of actions available for each player, and the corresponding payoff functions for each player respectively. Similar with a multi-objective optimization problem each player seeks to maximize her corresponding payoff function in order to increase her profits.

The most common solution concepts for the two types of problems are the Pareto optimal solution for MOOPs [5] and the Nash equilibria for games [10], respectively. While the Pareto dominance relation allowed an extensive study of MOOPs from an evolutionary perspective [4], for the Nash equilibrium only recently an appropriate fitness concept based on non-domination has been developed [6, 7]. This was attained with the use of a generative relation capable of guiding an evolutionary algorithm towards a game's Nash equilibrium. Thus existing evolutionary algorithms designed for multiobjective optimization can be adapted for searching Nash Equilibria.

Our aim is to compute equilibria for games involving large number of players. Preliminary experiments with [6, 3] show that common algorithms

Received by the editors: August 20, 2010.

2000 *Mathematics Subject Classification.* 91A80, 68T01.

1998 *CR Categories and Descriptors.* I.2.8 **Computing methodologies**: Artificial Intelligence *Problem Solving, Control Methods, and Search - Heuristic Methods*.

Key words and phrases. Nash equilibrium, generative relation for equilibria, large non-zero games.

designed for MOOPs are unsuccessfully answering that challenge. A memetic algorithm that aims to combine evolutionary techniques and local improvement procedures, named Global Search and Local Ascent (GSLA) algorithm, is introduced. Numerical experiments conducted for Cournot games with up to 100 players show the potential of GSLA.

The paper is organized as follows: in the next section some basic notions from game theory are presented as well as the Nash generative relation. In the third section the GSLA algorithm is described, and the algorithm's founding ideas and building blocks are illustrated. Next, numerical experiments conducted with the Cournot oligopoly offer an assessment of GSLA's performance, as well as a comparison to NSGA-II's. The paper ends with conclusions, acknoledgments and bibliographical references.

2. Prerequisites

Some fundamental concepts related to game theory are described in this section.

2.1. Strategic game. A finite strategic non-cooperative game [6, 10] is defined as a system $\Gamma = ((N, S_i, u_i), i = \overline{1, n})$ where:

- $N = \{1, ..., n\}$ is a set of n players;
- for each player $i \in N$, S_i represents the set of actions (pure strategies) available to him, $S_i = \{s_{i_1}, s_{i_2}, ..., s_{i_{m_i}}\}$;
- $S = S_1 \times S_2 \times ... \times S_N$ is the set of all possible situations of the game;
- an element of S is a strategy profile (or strategy) of the game;
- for each player $i \in N$, $u_i : S \to R$ represents the payoff function.

Let s^* be a strategy profile. Denote by (s_{i_j}, s^*_{-i}) the strategy profile obtained from s^* by replacing the strategy of player i by s_{i_j} i.e.

$$(s_{i_j}, s^*_{-i}) = (s^*_i, s^*_2, ..., s^*_{i-1}, s_{i_j}, s^*_{i+1}, ..., s^*_n).$$

S_{-i} denotes a strategy profile of every player except i. It is important to notice that, for a given game outcome, each player's payoff is not determined solely by his own action, but rather by the combination of his chosen strategy and all the other players' actions.

2.2. Nash Equilibrium. In game theory, the prevalent solution concept of a non-cooperative game is Nash Equilibrium (NE) [9, 8]. Thus, a Nash equilibrium is a strategy profile reflecting a state of the game from which no single player can improve his payoff by unilaterally modifying her strategy.

Definition Profile strategy s^* is a Nash equilibrium if the inequality $u_i(s^*) \geq u_i(s_{i_j}, s^*_{-i})$ holds for every action s_{i_j} of every player i, $s_{i_j} \in S_i$.

2.3. **Nash ascendancy relation.** Given two strategy profiles s' and s'', the instituted operator $k(s', s'')$ assigns to the pair the cardinality of the set

$$k(s'', s') = card\{i \in \{1, ..., n\} | u_i(s'_i, s''_{-i}) \geq u_i(s''), s'_i \neq s''_i\}$$

i.e. $k(s', s'')$ denotes the number of individual strategies from s' which replaced in s'' give better payoff for the corresponding player.

$k(s'', s')$ measures the sensitivity of s'' with respect to perturbations supplied from s'. The lower sensitivity, the higher is the stability of s'' with respect to s'.

We may use

$$m(s'', s') = n - k(s'', s')$$

as a measure for the relative quality of s'' with respect to s'.

Let us consider a generative relation R_N on $S \times S$: $(s', s'') \in R_N$ if and only if s' is better than s'' with respect to m, i.e. $m(s', s'') > m(s'', s')$.

Therefore $(s', s'') \in R_N$ if and only if $k(s', s'') < k(s'', s')$.

A strategy profile s' ascends (dominates in Nash sense) another strategy profile s'' if there are less players capable of augmenting their profit by changing their strategy from s' to s'', than the reverse.

A strategy profile s is considered non-dominated in Nash sense if $\nexists s' \in S$: $(s', s) \in R_N$.

As stated in [6], all non-dominated strategy profiles with respect to R_N represent NE.

3. Proposed method

The above-mentioned Nash-based domination concept facilitates the comparison of two solutions, ascertaining that one is "closer" than the other to the equilibrium. Applying this domination concept in the framework of evolutionary computation, by using the generative relation within the EA comparison procedures, leads to algorithms for search and detection of a game's Nash equilibria, as the algorithm will converge to the Nash non-dominated solutions. An evolutionary model that incorporates a global search within the game solutions' space is proposed. This search is performed using a genetic algorithm that has been adapted so it detects a game's Nash equilibrium. Then, a local search algorithm is used, aimed to improve the quality (thus reducing the "distance" to the NE) of the new population's best candidate solution.

The proposed method, named Global Search and Local Ascent (GSLA), is constructed as a memetic algorithm, a GA's hybridization with a stochastic local optimization technique.

3.1. **Global Search.** Using a Genetic algorithm (GA) [1, 4] a population of individuals is evolved, by applying a set of particular rules, towards a state which maximizes the population's fitness. Because of the many advantages inherent to this technique, as well as its compatibility with the problem of detecting a game's NE, the model of a GA constitutes the basis of the proposed evolutionary method.

The algorithm uses a real-coding of chromosomes: each chromosome represents a strategy profile and each individual gene indicates the strategy that a player chooses to play.

In its run, the algorithm first triggers the creation of a random initial population of candidate solutions. Each generation's individuals are assigned – via a Nash-ascendancy fitness evaluation – an factor that indicates the quality of the candidate's solution within the population. Less dominated individuals (with a smaller ascendancy factor) represent better-quality solutions.

Selection of individuals for crossover is performed tournament-style offering the benefit of a stricter selection pressure while still not completely discounting weaker individuals. Selected individuals undergo a convex crossover, their offspring possibly also undergo uniform mutation to preserve genetic variety within the population.

The offspring population, with the same size with the parent population, replaces the old population, discarded in favour of the young one. The best individual of the old population is also kept in the new population – a partial elitist approach which avoids the risk of premature convergence associated with full elitism, but still ensures the further exploration of the current best solution and its neighbours within the next generation. The algorithm stops after a preset number of generations or when the number of fitness function evaluations is reached.

3.2. **Local search.** GAs are usually competent at detecting adequate global solutions, but are often lacking precision. On the other hand, local search methods, such as hill climbing, are quite successful at detecting the optimum in a bounded section. An approach wherein the GA technique and hill climbing are alternated should increase the search' efficiency while surmounting the deficiencies inherent in GA as well as in hill climbing.

Within the proposed model, after reaching a specified number of iterations, before resuming with the initiation of a new generation cycle, the current generation's fittest individual is in fact set apart and submitted to undergo a local refinement process (before being added to the new population).

This process is constructed as follows:

Let $s = (s_1, ..., s_n)$ denote the strategy profile represented by the fittest individual. For a random value $i \in \{1, ..., n\}$, the gene s_i is modify by summing a value $\pm e$ ($\pm$ is randomly decided upon). For the potential offspring s', we would have $s'_i = s_i \pm e$.

If the new strategy profile Nash-ascends its parent, it takes its place and the refinement process is applied again.

If the new strategy profile does not dominate its parent a new position i is considered for mutation. The following positions are obtained by either increases or decreases (a randomly made decision as well) the precedent i value by 1, depicting a circular movement along the chromosomal genes.

If, however, within the current strategy profile s no i was found so that player s' could increase her payoff by altering her strategy with e, the search ends.

4. Numerical experiments

The current section offers a brief view into the performance analysis of the GSLA algorithm, as well as a performance comparison with multi-objective optimization evolutionary algorithm NSGA-II [3], adapted to search for a game's Nash equilibria instead of the Pareto optimal solutions.

The Cournot oligopoly model has been considered for the numerical experiments [2]. Results show the two algorithms – GSLA and NSGA-II – individual performances in detecting the game's NE. A graphical visualization of the obtained values also illustrates how the two algorithms behave.

4.1. **Cournot oligopoly.** Consider n competing companies, all producing a single product, and the product quantity produced by each firm is denoted by q_i respectively.

A game strategy profile is

$$s = (q_1, ..., q_n).$$

Let $Q = \Sigma q_i$ denote the total quantity for that product available on the market. The market clearing price is

$$P(Q) = \begin{cases} a - Q, & \text{for } Q < a, \\ 0, & \text{for } Q \geq a. \end{cases}$$

where a denotes the maximum number of the product that are possible to be sold on the market.

We assume that any firm i's full expenditure for producing the quantity q_i is $C_i\ (q_i) = cq_i$, $c < a$. Working with the supposition that all firms choose their quantities at the same time, the payoff for each firm i is its profit:

$$\begin{aligned} \pi_i(q_1, ..., q_n) &= q_i P(Q) - C_i(q_i) \\ &= q_i \left[a - (q_1 + ... + q_n) - c\right], i = 1, ..., n. \end{aligned}$$

	GSLA		NSGA II	
No. of players	Average distance	Standard deviation	Average distance	Standard deviation
2	0.000074	0.000050	0.000757	0.001712
5	0.000548	0.001124	1.321731	1.758642
10	0.002871	0.004088	7.467808	2.818299
20	0.265021	0.230464	18.976990	4.420855
50	0.379441	0.256289	26.378490	3.942610
100	0.447361	0.029569	89.040740	16.546090

Table 1. Average distance to Nash equilibrium in 30 runs for GSLA and NSGA II

The Cournot oligopoly in the form presented here has one Nash equilibrium $q^* = (q_i{}^*)_{(i=\overline{1,n})}$, and

$$q_i{}^* = \frac{a-c}{n+1}, \forall i \in \{1, ..., n\}$$

This model is used for numerical simulation with different numbers of players in order to asses GSLA's and NSGA-II's performances in detecting the game's Nash equilibrium.

4.2. **Parameter setting.** Both algorithms GSLA and NSGA-II are tested for 2, 5, 10, 20, 50 and 100 players.

Population size was set to 75, the maximum number of generations to 250, the initialization domain to [0, 100]. The tournament size is 30, crossover probability 0.5 and probability of mutation 0.2.

Cournot parameters a and c were set to 205 and 3 respectively.

30 runs were completed for each algorithm with different random seeds, and the average and standard deviation of the best obtained solutions' distance to the NE was calculated.

4.3. **Numerical results.** The numerical results obtained for games with 2, 5, 10, 20, 50 and 100 players with the two methods – GSLA and NSGA II are depicted in Table 1. The GSLA performance is superior, the average distance to Nash equilibria being 0.44 even for the game having 100 players while NSGA-II has an average distance 89.04. A graphical illustration of the differences between the two methods is given in Figure 1.

5. Conclusion and Further work

A new hybrid method, called Global Search and Local Ascent algorithm is proposed. GSLA combines a generational evolutionary algorithm with a hill

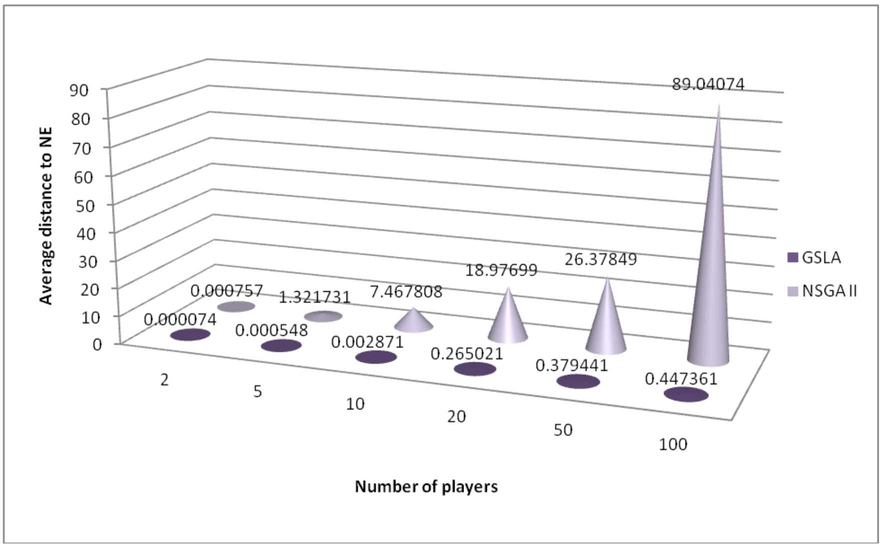

Figure 1. Comparison of average distance to NE for GSLA and NSGA II

climbing procedure in order to compute the Nash equilibria for a large non-cooperative game. The search is guided using a generative relation allowing the comparison of two strategy profiles within a game.

The efficiency of the method is evaluated using a Cournot oligopoly taking into account up to 100 firms.

Results are compared with a modify version of the NSGA-II algorithm. For the given setting, GSLA significantly outperforms NSGA-II, suggesting a very good search potential.

Further work will consist in exploring this potential by using GSLA for equilibria detection in games characterized by the existence of multiple Nash equilibria.

6. Acknowledgements

This research is supported partially from the Sectorial Operational Programme Human Resources Development, Contract POSDRU 6/1.5/S/3 "Doctoral studies: through science towards society", Babeş - Bolyai University, Cluj - Napoca, România, and by CNCSIS UEFISCSU, project number PNII IDEI 2366/2008, and by the CNCSIS Grant ID508 *"New Computational paradigms for dynamic complex problems"* funded by the MEC.

References

[1] T. Bäck, *Evolutionary Algorithms in Theory and Practice: Evolution Strategies, Evolutionary Programming, Genetic Algorithms*, Oxford Univ. Press, 1996.

[2] A. F. Daughety, *Cournot oligopoly: characterization and applications*, edited by Andrew. F. Daughety, Cambridge University Press, Cambridge; New York, 1988.

[3] K. Deb, S. Agrawal, A. Pratap, T. Meyarivan, *A fast elitist non-dominated sorting genetic algorithm for multi-objective optimization: NSGA-II* , Springer, 2000, pp. 849-858.

[4] D. E. Goldberg,*Genetic Algorithms in Search Optimization and Machine Learning*, Addison Wesley, 1989.

[5] H. Ishibuchi, N. Tsukamoto, Y. Nojima, *Evolutionary many-objective optimization: A short review*, in Proc. IEEE Congr. Evol. Comput., Hong Kong, Jun. 2008, pp. 2424-2431.

[6] R. I. Lung, D. Dumitrescu, *Computing Nash equilibria by means of evolutionary computation*, in Int. J. of Computers, Communications & Control, vol. III, no. suppl. issue, 2008, pp. 364-368.

[7] R. I. Lung, T. D. Mihoc, D. Dumitrescu, *Nash Equilibria Detection for Multi-Player Games*, WCCI 2010 IEEE World Congress on Computational Intelligence, July, Barcelona, Spain, 2010, pp.3447-3451.

[8] R. D. McKelvey, A. McLennan, *Computation of Equilibria in Finite Games*, in Handbook of Computational Economics, H. M. Amman, D. A. Kendrick, J. Rust, Eds. Elsevier, Amsterdam, 1996, vol. 1, ch. 2, pp. 87-142.

[9] J. F. Nash, *Non-cooperative games*, in The Annals of Mathematics, Second Series, vol. 54, issue 2, 1951, pp. 286-295.

[10] V. V. Vazirani, N. Noam, T. Roughgarden,É. Tardos,*Algorithmic Game Theory*, Cambridge University Press, Cambridge, 2007.

(1) Babeş-Bolyai University, Faculty of Mathematics and Computer Science, 1 Mihail Kogălniceanu St., RO-400084 Cluj-Napoca, Romania

(2) Babeş-Bolyai University, Faculty of Economics and Business Administration, 58-60 Teodor Mihaly St., RO-400591 Cluj-Napoca, Romania

E-mail address: smsd0362@scs.ubbcluj.ro

E-mail address: mihoct@cs.ubbcluj.ro

E-mail address: rodica.lung@econ.ubbcluj.ro

E-mail address: ddumitr@cs.ubbcluj.ro