# NONMONOTONIC SKEPTICAL CONSEQUENCE RELATION IN CONSTRAINED DEFAULT LOGIC

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ABSTRACT. This paper presents a study of the nonmonotonic consequence relation which models the skeptical reasoning formalised by constrained default logic. The nonmonotonic skeptical consequence relation is defined using the sequent calculus axiomatic system. We study the formal properties desirable for a "good" nonmonotonic relation: supraclassicality, cut, cautious monotony, cumulativity, absorption, distribution.

## 1. INTRODUCTION

*Default logics* [10] represent a simple but a powerful class of nonmonotonic formalisms. These logical systems capture and model *defeasible inference*, a type of inference which permits that in the light of new information, already derived conclusions, called *beliefs*, to be retracted. The corresponding reasoning process is not a monotonic one: the set of derived conclusions does not increase by adding new premises.

The family of default logics represent information using a *default theory* $(D, W)$, containing a set $W$ of first-order formulas, called *facts* (explicit information) and a set $D$ of inference rules, called *defaults* (implicit information). These special inference rules model laws that are true with a few exceptions. According to [7] a default has the syntax: $d = \frac{\alpha:\beta}{\gamma}$, where $\alpha, \beta, \gamma$ are formulas of first-order logic, $\alpha$ is the *prerequisite*, $\beta$ is the *justification* and $\gamma$ is the *consequent*. The default $d = \frac{\alpha:\beta}{\gamma}$ can be applied and thus derive $\gamma$ if $\alpha$ is believed and it is consistent to assume $\beta$.

The differences among the variants (classical, justified, constrained, rational) of default logic are caused by the semantics of the defaults. The defaults extend a given set of facts obtaining one or more sets called *extensions* which

contain the *nonmonotonic consequences* (beliefs). The reasoning process can be viewed as a process of inferring consequences of both explicit and implicit content of the knowledge base (default theory).

The extensions represent possible belief sets of an agent reasoning about the initial default theory. A *credulous reasoning* perspective means that an agents' beliefs belong to at least one extension. *Skeptical consequences* are more robust beliefs because they belong to all extensions of a theory.

The nonmonotonic reasoning can be approached from an abstract point of view, like deductive reasoning was studied using the classical inference relation/operation [1, 3, 6, 11]. The properties of a nonmonotonic inference relation/operation specific to default logics characterise the influence of the set of facts (explicit content of the knowledge base) on the nonmonotonic reasoning process.

In the papers [2, 4, 5, 8] the credulous/skeptical nonmonotonic reasoning modeled by different nonmonotonic logics, including default logics, is described using sequent calculi-based axiomatic systems.

From all versions of default logic we have chosen to study constrained default logic [9] because it has the most desirable computational properties (semi-monotonicity, commitment to assumption) regarding the application of defaults. We will prove that this version of default logic also satisfies important formal properties regarding the influence of facts on the reasoning process.

Based on the *skeptical constrained default sequent calculus* axiomatic system introduced in [5], in this paper we define the *nonmonotonic consequence relation* which models the *skeptical reasoning* formalised by constrained default logic. The formal properties desirable for a "good" nonmonotonic relation: *consistency preserving, supraclassicality, cut, cautious monotony, cumulativity, absorption, distribution*, are studied.

## 2. Nonmonotonic inference relations

The authors of [1, 6, 11] classified and enumerated the properties of a nonmonotonic inference operation: pure conditions, relations with the classical consequence relation, interaction with logical connectives.

A nonmonotonic inference relation denoted by "$S \mid\sim \varphi$", is defined between a set S of formulas (the premises of the inference) and a formula (the consequence of the inference).

In the following we will express the above properties in a relational manner, corresponding to a nonmonotonic (inference) consequence relation.

The *pure conditions* contain the properties inherited from the classical consequence relation ($\vdash$): *reflexivity, transitivity*, and specific properties: *cut, cautious monotony, cumulativity*, which try to weaken the monotony property

of the deductive classical relation (if $S \vdash \varphi$ and $S \subseteq V$ then $V \vdash \varphi$ ) which is given-up in a nonmonotonic reasoning.

- **Reflexivity**: if $\varphi \in S$ then $S \mid\sim \varphi$.
  All formulas of $S$ are nonmonotonic consequences of $S$.
- **Transitivity**: if $S \mid\sim \varphi$ and $\{\varphi\} \mid\sim \psi$ then $S \mid\sim \psi$.
- **Cut**: if $S \mid\sim \varphi$ and $S \cup \{\varphi\} \mid\sim \psi$ then $S \mid\sim \psi$.
  Adding to a set $S$ a formula which is already a nonmonotonic consequence of $S$ does not lead to any increase in inferential power.
- **Cautious monotony**: if $S \mid\sim \varphi$ and $S \mid\sim \psi$ then $S \cup \{\varphi\} \mid\sim \psi$.
  *Cautious monotony* is the converse of *cut* and has the meaning: the information derived during the reasoning process and added to the set of premises does not decrease the set of nonmonotonic consequences.
- **Cumulativity**: cut + cautious monotony:
    if $S \mid\sim \varphi$ then:
        $S \mid\sim \psi$ if and only if $S \cup \{\varphi\} \mid\sim \psi$.
  *Cumulativity* permits the addition of lemmas to the set of premises without affecting the inferential process.
- **Reciprocity**:
    if $\forall \varphi \in V : S \mid\sim \varphi$ and $\forall \varphi \in S : V \mid\sim \varphi$ then:
        $S \mid\sim \psi$ if and only if $V \mid\sim \psi$.

The following properties characterise the relationships between the non-monotonic inference relation ($\mid\sim$) and the classical inference relation, operation ($\vdash$, $Th(S) = \{\varphi | S \vdash \varphi\}$) and also define the interactions with the logical connectives in classical logic.

- **Supraclassicality**: if $S \vdash \varphi$ then $S \mid\sim \varphi$.
  A monotonic classical consequence of a set $S$ of premises is also a nonmonotonic consequence of the same set $S$.
- **Distribution**: if $S \mid\sim \varphi$ and $V \mid\sim \varphi$ then $Th(S) \cap Th(V) \mid\sim \varphi$.
- **Left logical equivalence**:
    if $Th(S) = Th(V)$ then: $S \mid\sim \varphi$ if and only if $V \mid\sim \varphi$.
- **Right weakening**: if $S \mid\sim \varphi$ and $\{\varphi\} \vdash \psi$ then $S \mid\sim \psi$.
  Right weakening is a weak transitivity.
- **Subclassical cumulativity**:
    if $S \subseteq V$ and $(\forall \varphi \in V : S \vdash \varphi)$ and $S \mid\sim \psi$ then $V \mid\sim \psi$.
  This property is a "weak" monotony: if we add only formulas monotonically derived from the premises to the set of premises, the number of nonmonotonic consequences may increase.
- **Left absorption**: $S \mid\sim \psi$ if and only if $V \vdash \psi$ , when $\forall \varphi \in V : S \mid\sim \varphi$.
- **Right absorption**:
    $S \mid\sim \psi$ if and only if $V \mid\sim \psi$ ,when $\forall \varphi \in V : S \vdash \varphi$ .

- **Full absorption**: left absorption + right absorption.
- **Right "and"**: if $S \mid\sim \varphi$ and $S \mid\sim \psi$ then $S \mid\sim \varphi \wedge \psi$ .
- **Left "or"**: if $S \cup \{\varphi\} \mid\sim \lambda$ and $S \cup \{\psi\} \mid\sim \lambda$ then $S \cup \{\varphi \vee \psi\} \mid\sim \lambda$ .
- **Conditionalization**: if $S \cup \{\varphi\} \mid\sim \psi$ then $S \mid\sim \varphi \rightarrow \psi$.
- **Proof by cases**: if $S \cup \{\varphi\} \mid\sim \psi$ and $S \cup \{\neg\varphi\} \mid\sim \psi$ then $S \mid\sim \psi$ .

The following relations between the above properties hold:
1. supraclassicality + cumulativity $\Longrightarrow$ full absorption;
2. reflexivity + reciprocity $\Longleftrightarrow$ cumulativity;
3. left absorption $\Longrightarrow$ right "and", right weakening;
4. right absorption $\Longrightarrow$ left logical equivalence, subclassical cumulativity;
5. distribution + supraclassicality + absorption $\Longrightarrow$
    left "or", proof by cases, conditionalization;

The monotonicity is given-up in a defeasible reasoning and according to [1, 7, 11] the following properties are natural and desirable for a nonmonotonic consequence relation: *consistency preserving + supraclassicality + cumulativity + distribution.*

## 3. Skeptical constrained default sequent calculus

A specific sequent calculus, based on classical sequent/anti-sequent calculi enhanced with residues is used to express the skeptical constrained default reasoning [5].

For a set $D$ of defaults we define the set of *residues* and the set of *justifications of D with respect to the set C* of formulas, using the classical anti-sequent calculus (metasybol: $\not\Rightarrow$) as follows:

$Res_D^C = \{\frac{\alpha}{\gamma} \mid \frac{\alpha:\beta}{\gamma} \in D, C \cup \{\beta, \gamma\} \not\Rightarrow false\}$,

$Justif_D^C = \{\beta \mid \frac{\alpha:\beta}{\gamma} \in D, C \cup \{\beta, \gamma\} \not\Rightarrow false\}$.

The residues corresponding to the applied defaults are monotonic rules and are used to reduce the nonmonotonic reasoning process modeled by constrained default logic into a monotonic one according to the following theorem.

**Theorem 1**[5]: Let $\Delta = (D, W)$ be a default theory. $(E, C)$ is a *constrained extension* of $\Delta$ if $E = Th^{res}(W, Res_D^C)$ and $C = Th(Th^{res}(W, Res_D^C) \cup Justif_D^C)$, where $Th(\cdot)$ is the classical consequence operator and $Th^{res}(\cdot, R)$ is the consequence operator of the propositional formal system enhanced with the set $R$ of residues. $E$ is the *actual extension* embedded in the *reasoning context C*.

The **sequent/anti-sequent rules for residues** use the same metasymbols $(\Rightarrow, \not\Rightarrow)$ like classical logic:

(Re1)$\frac{\Gamma\Rightarrow\Psi}{\Gamma,\frac{\alpha}{\gamma}\Rightarrow\Psi}$;     (Re2) $\frac{\Gamma\Rightarrow\alpha\quad\Gamma,\gamma\Rightarrow\Psi}{\Gamma,\frac{\alpha}{\gamma}\Rightarrow\Psi}$;     (Re3) $\frac{\Gamma\not\Rightarrow\alpha\quad\Gamma\not\Rightarrow\Psi}{\Gamma,\frac{\alpha}{\gamma}\not\Rightarrow\Psi}$;     (Re4)$\frac{\Gamma,\gamma\not\Rightarrow\Psi}{\Gamma,\frac{\alpha}{\gamma}\not\Rightarrow\Psi}$.

Let $\Delta = (D, W)$ be a default theory. A *skeptical constrained default sequent* has the syntax: $Constr; (W, D); Res \longmapsto U$. The set $U$ of formulas is called *succedent*. The *antecedent* contains $Constr$ (a set of constraints expressed using the modalities: M-possibility and L-necessity), the default theory $(W, D)$ and $Res$ (the set of residues corresponding to the applied defaults).

The skeptical reasoning formalized by constrained default logic is described in [5] using the *skeptical constrained default axiomatic system*:

$Sk_\Delta^{cons} = \left(\Sigma_{Sk_\Delta}^{cons}, F_{Sk_\Delta}^{cons}, A_{Sk_\Delta}^{cons}, R_{Sk_\Delta}^{cons}\right)$ , where $\Delta = (D, W)$ and:

$\Sigma_{Sk_\Delta}^{cons}$ is the alphabet;

$F_{Sk_\Delta}^{cons}$ contains all classical sequents/anti-sequents enhanced with residues and all skeptical constrained default sequents as below.

$A_{Sk_\Delta}^{cons}$ = the axioms (all classical basic sequents and anti-sequents).

$R_{Sk_\Delta}^{cons}$ - the classical sequent/anti-sequent rules, the rules for residues (Re1, Re2, Re3, Re4) and the sequent rules for skeptical constrained logic(S1, S2, S3) from below.

### Sequent rules for skeptical constrained default logic:

(S1) $\frac{Constr^M \cup W \not\Rightarrow false \quad W \cup Res \Rightarrow U}{Constr;(W,D);Res \longmapsto U}$, $Constr^M = \{\alpha | M\alpha \in Constr\}$;

(S2) $\frac{Constr \cup \{M(\beta \wedge \gamma)\};(W,D);Res \cup \{\frac{\alpha}{\gamma}\} \longmapsto U \quad Constr \cup \{L\neg(\beta \wedge \gamma)\};(W,D);Res \longmapsto U}{Constr;(W,D \cup \{\frac{\alpha:\beta}{\gamma}\});Res \longmapsto U}$

(S3) $\frac{W \cup \{\beta \wedge \gamma | \frac{\alpha:\beta}{\gamma} \in D\} \not\Rightarrow \delta}{Constr \cup \{L\delta\};(W,D);Res \longmapsto U}$

The above rules are based on the properties: semimonotonicity, commitment to assumption and the fact that the nonmonotonic reasoning process modelled by constrained default logic is guided by a maximal consistent reasoning context.

**Theorem 2**[5]: A formula X is a skeptical constrained default consequence of the default theory $\Delta = (D, W)$ if and only if the skeptical constrained default sequent $\emptyset; (W, D); \emptyset \longmapsto X$ is true (can be reduced to basic sequents/anti-sequents using $Sk_\Delta^{cons}$).

## 4. Nonmonotonic skeptical default consequence relation for constrained default logic

Based on Theorem 2 we define the nonmonotonic skeptical constrained default consequence relation $|\sim_s^c$ and we will study its formal properties.

Let $\Delta = (D, W)$ be a default theory and $X$ a formula:

$(D, W) |\sim_s^c X$ if the sequent $\emptyset; (W, D); \emptyset \longmapsto X$ is true using $Sk_\Delta^{cons}$.

The formal properties satisfied by the nonmonotonic skeptical constrained default consequence relation $|\sim_s^c$ are established in the following theorem.

**Theorem 3**:

(1) Let $\Delta = (D, W)$ be a default theory with no restrictions imposed. The formal properties satisfied by $|\sim_s^c$ are as follows:
- **Consistency preserving**: the skeptical default reasoning based on a consistent set of facts will not introduce contradictions.
- **Reflexivity**: if $X \in W$ then $(D, W) |\sim_s^c X$.
  The facts are skeptical default consequences of the theory $\Delta$.
- **Supraclassicality**: if $W \vdash X$ then $(D, W) |\sim_s^c X$.
  Constrained default logic extends the classical logic from the inferential point of view.
- **Cut**:
  if $(D, W) |\sim_s^c X$ and $(D, W \cup \{X\}) |\sim_s^c Y$ then $(D, W) |\sim_s^c Y$.
- **Full absorption**= *left absorption + right absorption*:
  **- left absorption:**
  $(D, W) |\sim_s^c X$ if and only if $V \vdash X$, when $\forall Y \in V, (D, W) |\sim_s^c Y$.
  **- right absorption:**
  $(D, W) |\sim_s^c X$ if and only if $(D, W) |\sim_s^c X$, when $\forall Y \in V, W \vdash Y$.
  This property is specific to the logical approaches used to formalize the nonmonotonic reasoning and it is not satisfied by procedural approaches.
- the properties derived from absorption: **right weakening, right and, left logical equivalence, subclassical cumulativity**.

(2) Let $\Delta = (D, W)$ be a default theory with $D$ containing only defaults free of prerequisites $\frac{:\beta}{\gamma}$. The properties from (1) are satisfied and also:
- **Cumulativity**: **cautious monotony + cut**:
  if $(D, W) |\sim_s^c X$ then:
  $(D, W) |\sim_s^c Y$ if and only if $(D, W \cup \{X\}) |\sim_s^c Y$.
  This property is an alternative of monotony in nonmonotonic formalisms and permits the use of lemmas (nonmonotonic consequences already derived) in the reasoning process.

(3) Let $\Delta = (D, W)$ be a default theory with $D$ containing only normal defaults free of prerequisites $\frac{:\beta}{\beta}$. The properties from (1), (2) are satisfied and also:
- **Reciprocity**:
  if $\forall X \in V, (D, W) \mid\sim_s^c X$ and $\forall X \in W, (D, V) \mid\sim_s^c X$ then:
  $(D, W) \mid\sim_s^c Y$ if and only if $(D, V) \mid\sim_s^c Y$.
- **Distribution**: if $(D, W) \mid\sim_s^c X$ and $(D, V) \mid\sim_s^c X$ then
  $(D, Th(W) \cap Th(V)) \mid\sim_s^c X$.
- the derived properties: **proof by cases, left or, conditionalization**.

*Proof*: The above properties are easily proved using the sequent/antisequent rules for constrained default logics (S1, S2, S3), for residues (Re1, Re2, Re3, Re4) and the rules for classical logic.

The following examples present some negative results regarding the properties of nonmonotonic skeptical default consequence relation: *distribution* and *cumulativity*.

**Example 1**: Let $D = \{d_1 = \frac{a:c}{c}, d_2 = \frac{\neg a:c}{c}\}$ be a set of normal defaults with prerequisites. The property "proof by cases" is not satisfied, and thus neither distribution.

Using $Sk^{cons}$ we prove that $(D, \{a\}) \mid\sim_s^c c$, $(D, \{\neg a\}) \mid\sim_s^c c$, but $(D, \emptyset) \mid\not\sim_s^c c$.

$$
\cfrac{\cfrac{\{a,c\}\not\Rightarrow false \qquad \{a,\frac{a}{c}\}\Rightarrow c}{\{Mc\};(\{a\},\{d2\});\{\frac{a}{c}\} \longmapsto c}\, S1 \qquad \cfrac{\{a,c\}\not\Rightarrow \neg c}{\{L\neg c\};(\{a\},\{d2\});\emptyset \longmapsto c}\, S3}{\emptyset;(\{a\},\{d1=\frac{a:c}{c}, d2=\frac{\neg a:c}{c}\});\emptyset \longmapsto c}\, S2
$$

The above up-side-down binary tree represents the reduction of the skeptical default sequent: $\emptyset; (\{a\}, D); \emptyset \longmapsto c$, to two basic anti-sequents and a true sequent: $\{a, \frac{a}{c}\} \Rightarrow c$ which can be reduced further with $Re2$ to two basic sequents. Thus we have proved: $(D, \{a\}) \mid\sim_s^c c$.

In a similar manner we can prove that $c$ is also a skeptical default consequence of the default theory $(\{\neg a\}, D) : (D, \{\neg a\}) \mid\sim_s^c c$.

If we try to reduce the skeptical default sequent: $\emptyset; (\emptyset, D); \emptyset \longmapsto c$, we remark that the residues rule from the sequent $\{\frac{a}{c}\} \Rightarrow c$ (if we apply first $d_1$) cannot be applied because there are no facts (see the following reduction tree). Similarily the default $d_2$ cannot be apply.

$$
\cfrac{\cfrac{\{a,c\}\not\Rightarrow false \qquad \{\frac{a}{c}\}\Rightarrow c}{\{Mc\};(\emptyset,\{d2\});\{\frac{a}{c}\} \longmapsto c}\, S1 \qquad \cfrac{\{a,c\}\not\Rightarrow \neg c}{\{L\neg c\};(\{a\},\{d2\});\emptyset \longmapsto c}\, S3}{\emptyset;(\emptyset,\{d1=\frac{a:c}{c}, d2=\frac{\neg a:c}{c}\});\emptyset \longmapsto c}\, S2
$$

The initial skeptical default sequent cannot be reduced to basic sequents/anti-sequent and thus $(D, \emptyset) \not\mid\sim_s^c c$.

We conclude that the "proof by cases" (a particular case of distribution) property is not satisfied and neither distribution.

**Example 2** [6]: shows that the cautious monotony and cumulativity are not satisfied in skeptical reasoning formalized by constrained default logic for normal default theories having defaults with prerequisites.

Let $D = \{d_1 = \frac{:a}{a}, d_2 = \frac{a:b}{b}, d_3 = \frac{b:\neg a}{\neg a}\}$ be a set of normal defaults. $E1 = Th(\{a, b\})$ is the unique extension of $(D, \emptyset)$. We have that $(D, \emptyset) \mid\sim_s^c a$ and $(D, \emptyset) \mid\sim_s^c b$.

The default theory $(D, \{b\})$ has as constrained default extensions $E1 = Th(\{a, b\})$ and $E2 = Th(\{\neg a, b\})$. We remark that $a \notin E1 \cap E2$.

If we consider $b$ as a lemma, adding it to the initial set of facts will decrease the set of nonmonotonic skeptical consequences of the new default theory: $(D, \{b\}) \not\mid\sim_s^c a$.

Thus the cautious monotony is not satisfied and neither cumulativity.

Like in the previous example we can use $Sk^{cons}$ axiomatic system to prove $(D, \emptyset) \mid\sim_s^c a$, $(D, \emptyset) \mid\sim_s^c b$ and $(D, \{b\}) \not\mid\sim_s^c a$.

## 5. Conclusions

The nonmonotonic consequence relation which models the skeptical reasoning formalised by constrained default logic emphasises properties that characterises the reasoning process from an abstract point of view. Using the default sequent calculus axiomatic system for expressing the skeptical default reasoning we have studied properties inherited from classical logics and some specific properties.

According to the results from Section 4 we can conclude:

- For general default theories, the nonmontonic skeptical consequence relation extends (*supraclassicality, reflexivity*) and absorbs (*absorption*) the classical consequence relation. Adding a new fact, which is already a nonmonotonic consequence, to the default theory, does not lead to any increase in inferential power (*cut*).
- The normal default theories with defaults free of prerequisites represent a special class of theories, which have associated a nonmonotonic skeptical inference relation that satisfies the desirable properties: *absorption, cumulativity, distribution* and all the properties derived from them: *right weakening, right and, left logical equivalence, subclassical cumulativity, proof by cases, left or, conditionalization.*

- The existence of prerequisites of the defaults imposes an order in the application of the defaults and it is the cause of the lack of the properties: cautious monotony, cumulativity and distribution for the corresponding nonmonotonic skeptical inference relation.

All these properties are useful from the theoretical point of view and also to increase the efficiency in the computational process of obtaining the nonmonotonic skeptical constrained consequences of a default theory.

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