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ABOUT SELECTING THE "BEST" NASH EQUILIBRIUM

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Abstract. Most games simulating real-wold situations present multiple Nash equilibria. The problem of selecting one equilibrium is tackled using a generative relation for Nash equilibria. The equilibria ascending most strategies from a randomly generated population of strategies can be considered. Numerical examples are used to illustrate the method.

1. Introduction

The problem of selecting one equilibrium of a normal form game has been addressed in the literature in different ways. According to [1] there are three main approaches to deal with multiple Nash equilibria.

One is to introduce an equilibrium selection mechanism that specifies which equilibrium is picked up. Examples include random equilibrium selection, in [4], and the selection of an extremal equilibrium, as in [7].

The second approach is to restrict attention to a particular class of games, such as entry games, and search for an estimator which allows for identification of payoff parameters even if there are multiple equilibria. For example the models in [5, 6] and [3] study situations in which the number of firms is unique even though there may be multiple Nash equilibria. They propose estimators in which the number of firms, rather than the entry decisions of individual agents, is treated as the dependent variable.

A third method [12] Tamer, uses bounds to estimate an entry model. The bounds are derived from the necessary conditions for pure strategy Nash equilibria, which say that the entry decision of one agent must be a best response to the entry decisions of other agents. In [2] Berry and Reiss survey the econometric analysis of discrete games.

In this work a new selection method based on a generative relation for Nash equilibria for normal form games is proposed. The Nash ascendancy

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[8] relation can be used to compute Nash equilibria using Natural Computing methods such as Evolutionary Algorithms. In the case of multiple equilibria it can also be used to differentiate between them by determining which one ascends more strategies from a randomly generated population of strategies.

2. Nash ascendancy relation

A finite strategic game is defined by $\Gamma = ((N, S_i, u_i), i = \overline{1, n})$ where:

- N represents the set of players, $N = \{1, ..., n\}$, n is the number of players;
- for each player $i \in N$, S_i represents the set of actions available to him, $S_i = \{s_{i_1}, s_{i_2}, ..., s_{i_{m_i}}\}$ where m_i represents the number of strategies available to player i and $S = S_1 \times S_2 \times ... \times S_N$ is the set of all possible situations of the game;
- for each player $i \in N$, $u_i : S \to \mathbb{R}$ represents the payoff function.

Denote by (s_{i_j}, s^*_{-i}) the strategy profile obtained from s^* by replacing the strategy of player i with s_{i_j} i.e.

$$(s_{i_j}, s^*_{-i}) = (s^*_1, s^*_2, ..., s^*_{i-1}, s_{i_j}, s^*_{i+1}, ..., s^*_1).$$

The most common concept of solution for a non cooperative game is the concept of Nash equilibrium [9, 10]. A collective strategy $s \in S$ for the game Γ represents a Nash equilibrium if no player has anything to gain by changing only his own strategy.

Several methods to compute NE of a game have been developed. For a review on computing techniques for the NE see [9].

Consider two strategy profiles s^* and s from S. An operator $k : S \times S \to N$ that associates the cardinality of the set

$$k(s^*, s) = |(\{i \in \{1, ..., n\} | u_i(s_i, s^*_{-i}) \geq u_i(s^*), s_i \neq s^*_i\}|$$

to the pair (s^*, s) is introduced.

This set is composed by the players i that would benefit if - given the strategy profile s^* - would change their strategy from s^*_i to s_i, i.e.

$$u_i(s_i, s^*_{-i}) \geq u_i(s^*).$$

Let $x, y \in S$. We say the strategy profile x **Nash ascends** the strategy profile y in and we write $x \prec y$ if the inequality

$$k(x, y) < k(y, x)$$

holds.

Thus a strategy profile x dominates strategy profile y if there are less players that can increase their payoffs by switching their strategy from x_i to

y_i than vice-versa. It can be said that strategy profile x is more stable (closer to equilibrium) then strategy y.

Two strategy profiles $x, y \in S$ may have the following relation:

(1) either x dominates y, $x \prec y$ $(k(x,y) < k(y,x))$
(2) either y dominates x, $y \prec x$ $(k(x,y) > k(y,x))$
(3) or $k(x,y) = k(y,x)$ and x and y are considered indifferent (neither x dominates y nor y dominates x).

The strategy profile $s^* \in S$ is called non-ascended in Nash sense (NAS) if

$$\nexists s \in S, s \neq s^* \text{such that } s \prec s^*.$$

In [8] it is shown that all non-ascended strategies are NE and also all NE are non-ascended strategies. Thus the Nash ascendancy relation can be used to characterize the equilibria of a game.

3. Selection of Nash Equilibria

Using the ascendancy relation an equilibrium can be characterized by the number of strategies it ascends. The equilibrium ascending most strategy profiles may be considered to be the most "popular' equilibrium and thus a selection method is proposed.

In order to approximate the number of strategies ascended by an equilibrium the following method of comparing equilibria is proposed.

A population of strategies of size R is uniformly random generated 100 times. For each equilibrium and each population the number of strategies ascended by the equilibrium and the number of strategies that are indifferent to the equilibrium is computed.

The ratio of that number to R is a number between 0 and 1 representing a measure of ascendancy of that equilibrium. The average of these numbers over the 100 populations is denoted by M_a and can be used to compare different equilibria of a game.

The corresponding standard deviation S_a is also computed. These measures represent simple descriptive statistics tools that indicate the potential of the method.

4. Numerical examples

Several normal form games presenting multiple Nash Equilibria are presented. The size R is considered of 100000 strategies. Each game is presented by its payoff matrix and the Nash equilibria. For each NE, $M_a(NE)$ and $S_a(NE)$ are presented.

P1-P2	1	2
1	1,3	4,2
2	2,1	1,3

TABLE 1. Payoff table for Game 1

The payoff space illustrated for each game is generated by representing 100000 uniformly generated strategies. This representation is used in multi-objective optimization and offers some extra input in the features of the game.

4.1. **Game 1.** The first game has been chosen for illustration purposes. It is a game with two players each having two strategies. The payoff matrix is presented in Table 1. This game has one mixed NE at $(2/3, 1/3)$ and $(3/4, 1/4)$. Th e payoff space is visualized in Figure 1. All tested strategies are ascended by the NE of the game and $M_a(NE) = 100000$ and $S_a(NE) = 0$.

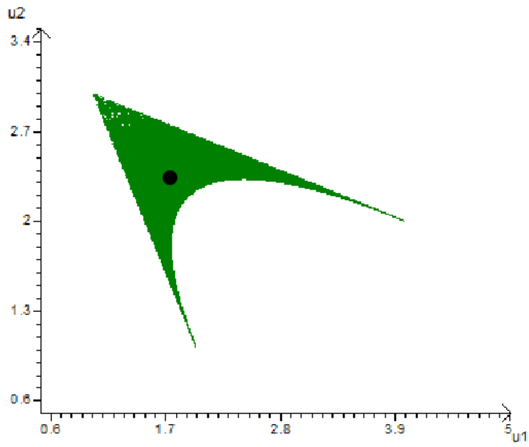

FIGURE 1. Game1. The circle represents the NE

4.2. **Game2.** This game represents a discrete four step version of the centipede game [11]. It is a two player game with two strategies for each player. The payoff matrix is presented in Table 2.

This game has two NE presented in Table 3, one in pure form and one in mixed form, both having the same payoff illustrated in Figure 2.

According to our experiments, both NE ascend the same number of strategies as $M_a(NE_1) = M_a(NE_2) = 1$ and $S_a(NE_1) = S_a(NE_2) = 0$. this is a specific feature for this game. Whatever the second player will chose, when

P1-P2	1	2
1	3,1	3,1
2	2,6	12,4

TABLE 2. Payoff table for Game 2

NE	1	2	Payoffs
1	1,0	1,0	(3,1)
2	1,0	0.9, 0.1	(3,1)

TABLE 3. NEs for Game 2

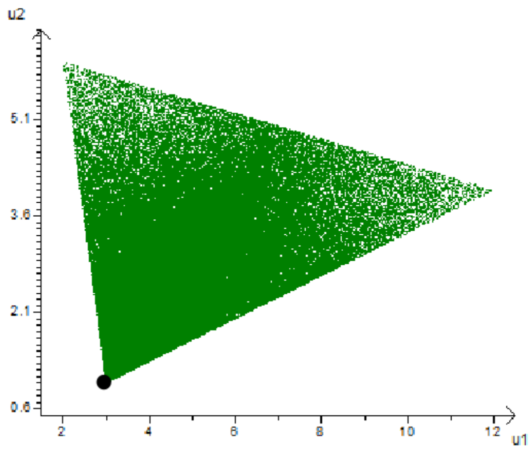

FIGURE 2. Game2. The circle represents the payoff for the two NE

the first one plays it first strategy the payoff for both is the same and there is no point in choosing between the two NE.

4.3. **Game 3.** The third game has two players each of them having three strategies available and payoffs in Table 4. It presents three Nash equilibria, two pure and one in mixed form as presented in Table 5. The pure Nash Equilibria weakly Pareto dominate the mixed one (Figure 3).

The results of our experiments are also presented in Table 5. These indicate a slightly 'higher popularity" of the mixed NE over the pure ones even though is weakly dominated by both. However, further statistical tools have to be used to determine if the difference between results is significant.

P1-P2	1	2	3
1	5,5	10,8	6,7
2	8,10	8,8	10,8
3	7,6	8,10	5,5

TABLE 4. Payoff table for Game 3

NE	1	2	Payoffs	M_a	S_a
1	**0.4, 0.6, 0**	**0.4,0.6,0**	**(8,8)**	**0.9904**	**0.0002**
2	1,0,0	0,1,0	(10,8)	0.9809	0.0004
3	0,1,0	1,0,0	(8,10)	0.9810	0.0004

TABLE 5. NEs for Game 3

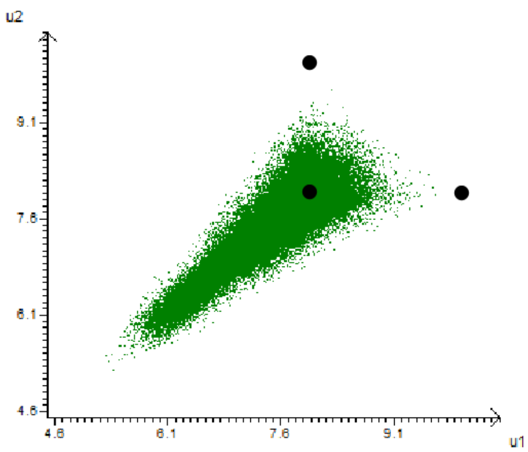

FIGURE 3. Payoff space for Game3. Circles represent payoffs of the three NE

4.4. **Game 4.** The fourth game is a two player game, each with three strategies. It presents nine NEs, three in pure form and six in mixed form. Payoffs are given in Table 6, the equilibria and results in Table 7 and the payoffs space is illustrated in Figure 4. According to these results, the 'best' choice would be the last NE yielding a payoff $(2, 2)$.

5. CONCLUSIONS AND FURTHER WORK

An attempt to introduce a new method for selecting between multiple Nash Equilibria of a normal form game is made in this paper.

P1-P2	1	2	3
1	3,1	0,0	0,1
2	1.5,1	2,2	1.5,1
3	0,1	0,0	3,1

TABLE 6. Payoff table for Game 4

NE	P1	P2	Payoffs	M_a	S_a
1	0.5,0.5,0	0.5714, 0.4285,0	(1.71,1)	0.7899	0.0011
2	0.5,0.5,0	0.5,0,0.5	(1.5,1)	0.8925	0.0009
3	0,0.5,0.5	0.5,0,0.5	(1.5,1)	0.8924	0.0009
4	0,0.5,0.5	0, 0.4285,0.5714	(1.71,1)	0.7898	0.0012
5	1,0,0	1,0,0	(3,1)	0.8670	0.0010
6	1,0,0	0.5,0,0.5	(1.5,1)	0.8670	0.0010
7	0,0,1	0.5,0,0.5	(1.5,1)	0.8669	0.0009
8	0,0,1	0,0,1	(3,1)	0.8669	0.0009
9	**0,1,0**	**0,1,0**	**(2,2)**	**0.9620**	**0.0005**

TABLE 7. NEs for Game 4

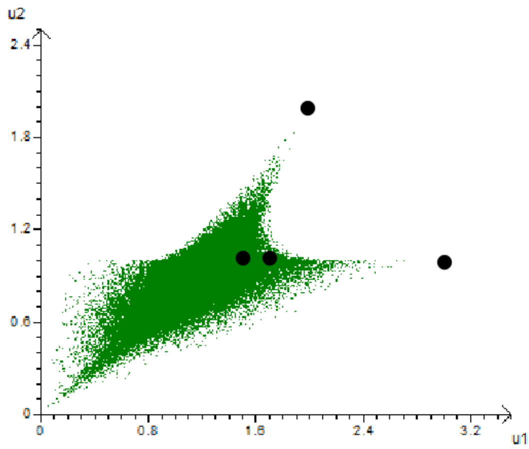

FIGURE 4. Payoff space for Game4. Circles represent payoffs of the NE

An ascendancy relation is used to determine which of the NE is most 'popular' or ascends most strategies from a randomly generated set. Some simple numerical experiments illustrate the use of this method.

Further work includes the use of statistical inference tools in order to evaluate the significance of the results.

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