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A NOTE ON A PROBLEM OF ŢÂMBULEA

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ABSTRACT. In [6] and [7] L. Ţâmbulea studied the number of positive integers sequences $(s_1, s_2, \dots, s_{2n+1})$ with the properties: $s_1 = s_{2n+1} = 1$ and $|s_i - s_{i+1}| = 1$ for each i between 1 and $2n$. In this note we discuss the algorithm to code this sequences by Dyck words and several related problems.

In [6] Ţâmbulea studied the problem of the number of elements in the set

$$S_n = \big\{(s_1, s_2, \dots, s_{2n+1}) \mid s_1 = s_{2n+1} = 1, |s_i - s_{i+1}| = 1 \text{ for } i = 1, \dots, 2n, \\ \text{and } s_j \in \mathbf{N}^* \text{ for } j = 2, 3, \dots, 2n\big\}$$

and proved that this is equal to the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. In [7] was proved that this number is equal to the number of possibilities to divide a convex $(n+2)$-gon into triangles using noncrossing diagonals. The equivalence with the problem of the number of binary trees with n nodes was proved too.

Stanley collected in [4, 5] more than a hundred problems related to Catalan numbers. This problem is not listed in [4] nor in [5]. In [5] there is a similar problem:

(p^5) Sequences $a_1, \dots, a_{2n}$ of nonnegative integers with $a_1 = 1, a_{2n} = 0$ and $a_i - a_{i-1} = \pm 1$.

Using the idea of coding elements of sets whose cardinality is a Catalan number, described in [1, 2], each sequence from S_n can be coded as follows. If $s_i - s_{i+1} = 1$ let us put a 0, and if $s_i - s_{i+1} = -1$ let us put a 1 in the code. E. g. the sequence 1,2,3,2,1 can be coded as 0011, and the sequence 1,2,1,2,1 as 0101. It is easy to see that such a code is a Dyck word, which is a binary word with equal number of 0s and 1s, the number of 1s never exceeding the number of 0s in each position from left to right. It is well-known that the number of $2n$-length Dyck words is the Catalan number C_n (see e.g. [8]). In the following algorithm description we consider correct inputs only.

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```
ENCODING1(s_1, s_2, ..., s_{2n+1})
j ← 0
for i ← 1 to 2n
    do j ← j + 1
        if s_{i+1} − s_i = 1
            then d_j ← 0
            else d_j ← 1
return d_1, d_2, ..., d_{2n}
```

```
DECODING1(d_1, d_2, ..., d_{2n})
s_1 ← 1; j ← 1
for i ← 1 to 2n
    do j ← j + 1
        if d_i = 0
            then s_j ← s_{j−1} + 1
            else s_j ← s_{j−1} − 1
return s_1, s_2, ..., s_{2n+1}
```

These two algorithms prove that between the set S_n and the set of $2n$-length Dyck words there is a bijection.

For the similar problem the codification can be made as follows. Consider the elements of sequence backwards. If $a_{i-1} - a_i = 1$ let us put a 0, else an 1 in the output word. At the end let us add a 1. For example, in the case of $n = 2$, we have two such sequences 1,2,1,0 and 1,0,1,0, the corresponding codes are: 0011 and 0101. Each code is a Dyck word. In the following algorithms we omitted the input verification.

```
ENCODING2(a_1, a_2, ..., a_{2n})
j ← 0
for i ← 2n downto 2
    do j ← j + 1
        if a_{i−1} − a_i = 1
            then d_j ← 0
            else d_j ← 1
d_{2n} ← 1
return d_1, d_2, ..., d_{2n}
```

```
DECODING2(d_1, d_2, ..., d_{2n})
a_{2n} ← 0;
j ← 2n
for i ← 1 to 2n
    do j ← j − 1
        if d_i = 0
            then a_j ← a_{j+1} + 1
            else a_j ← a_{j+1} − 1
return a_1, a_2, ..., a_{2n}
```

For a list of codifications of elements related to Catalan numbers see [3].

Another problem given in [7], which can be included in the set of problems being an interpretation of Catalan numbers, but not listed in [4, 5] is the following. A labelled binary tree is contructed by the rules:

- the root (which has the level 1) is labelled with 1,
- a node with the label i has a left subtree with the root labelled with $i-1$ and a right subtree with the root labelled with $i+1$,
- any subtree having the root labelled with 0 is omitted from tree.

The number of nodes labelled with 1 at level $2n+1$ is the Catalan number C_n. This can be proved easily by coding the paths between the root and nodes with label 1 at the level $2n+1$. Going from root on these paths, let us put 0 in code for a right edge and 1 for a left edge. In the Figure 1 for $n = 2$, we have two such paths with the codes: 0101 and 0011.

The labels of the nodes of such paths from root to the leaf correspond to the elements of sequence S_n, while the edges in the same order represent the bits of

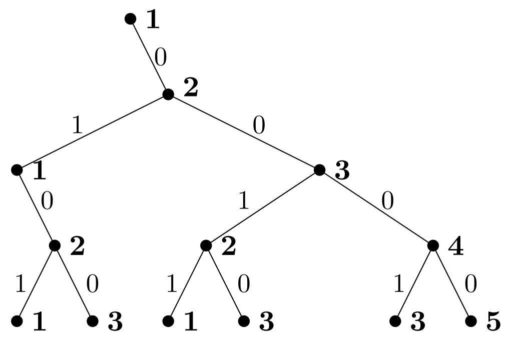

FIGURE 1. The tree corresponding to the set S_2. The two paths from the root to the nodes labelled with 1 at the level 5 are: 1,2,1,2,1 coded as 0101 and 1,2,3,2,1 coded as 0011.

the corresponding code. Such trees can be easily generated, so the generation of elements of S_n and the corresponding Dyck words is a straightforward task.

In future these two problems maybe will get into the list [5] of problems being an interpretation of Catalan numbers.

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