STUDIA UNIV. BABEŞ–BOLYAI, INFORMATICA, Volume **LIV**, Number 2, 2009

A PREDICTOR-CORRECTOR ALGORITHM FOR LINEARLY CONSTRAINED CONVEX OPTIMIZATION

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Abstract. In a recent paper we have introduced a new class of search directions for solving linear optimization (LO) problems. These directions are based on an algebraic equivalent transformation of the nonlinear equation from the system which defines the central path. However, from the implementation point of view predictor-corrector algorithms proved to be the most efficient among the class of interior point methods (IPMs). Therefore, we have defined also other variants of this class of algorithms, for example a weighted path-following algorithm, and a predictor-corrector algorithm for LO problems. Recently, the technique of finding search directions has been applied with success for linearly constrained convex optimization (LCCO), by Zhang, Bai and Wang. In this paper we define a new predictor-corrector algorithm for solving LCCO problems. We obtain new search directions by applying the method of algebraic equivalent transformation in this case too. Polynomial complexity of this algorithm is proved.

1. Introduction

The field of IPMs has been very active since Karmarkar published his famous paper [7] in 1984. However, in general one of the significant aspects of determining a new algorithm resides in the method of following the central path. Therefore, search directions play an important role in finding new algorithms. Peng, Roos and Terlaky [10] have defined the notion of self regular functions and, using this concept, they have introduced a new class of search directions for LO. They have extended their results also to complementarity problems (CP), semidefinite optimization (SDO) and second order cone optimization (SOCO), and they have proved polynomial complexity of

Received by the editors: November 20, 2009.

2010 *Mathematics Subject Classification.* 90C25, 90C51.

1998 *CR Categories and Descriptors.* G.1.6. [**Mathematics of Computing**]: Numerical Analysis – *Optimization* – *Convex programming*;

Key words and phrases. Linearly constrained convex optimization, interior-point methods, predictor-corrector algorithm, equivalent algebraic transformation, centering equation, Newton step.

different large-update algorithms, which use self-regular functions to obtain new directions.

An alternative method has been introduced in [3, 5] by applying algebraically equivalent transformations to the nonlinear centering equation of the system, which defines the central path. The method has been applied with success to convex quadratic optimization (CQO) by Achache [2] and LCCO by Zhang et al. [17]. Recently, the new technique for LO has been extended also to monotone mixed linear complementarity problems (LCPs) by Wang, Cai and Yue [15], and to SDO and SOCO by Wang and Bai [13, 14]. The method of algebraically equivalent transformation has been generalized also to weighted-path-following algorithms. The first results for LO have been given in [4]. Later on, Achache [1] generalized this algorithm to standard LCPs, and Wang et al. [16] to monotone horizontal LCPs. The above mentioned algebraic transformations, followed by a Newton step, resulted in small-update feasible algorithms, and for all of them the best known iteration bounds were obtained. However, Pan, Li and He [9] introduced a large-update infeasible algorithm using a logarithmic transformation. This logarithmic equivalent transformation was mentioned formerly by Tuncel and Todd [12].

Another approach of developing an efficient algorithm is considering predictor and corrector steps. We mention that the first algorithm, that have divided the Newton direction into the affine-scaling and centering direction, is due to Mehrotra [8]. In [6] we have defined a predictor-corrector algorithm obtained by applying an equivalent algebraic transformation, using the square root function, as in [3]. In this paper we extend this algorithm to LCCO.

The notations used in this paper are the following: $\Re^n$ is the set of n-dimentional vectors and $\Re^{m\times n}$ is the set of $m \times n$ matrices. Moreover, $\Re^+$ is the set of nonnegative real numbers, if $x \in \Re^n$ then $diag(x)$ is the diagonal matrix formed by the elements of x, and e is the all-one vector.

This paper is organized in the following way. In the next section we introduce the basic issues regarding the central path, and we discuss the primal-dual algorithm for LCCO. In the third section we define the predictor-corrector algorithm for LCCO and we prove its polynomial complexity. Finally, we present some conclusions in Section 4.

2. Primal-Dual Path-Following Algorithm

Let us consider the following problem

$$(P) \qquad min\ \{f(x) : Ax = b, x \geq 0\},$$

and its Wolfe dual

$$(D) \qquad max\ \{b^T y + f(x) - (\nabla f(x))^T x : A^T y + s - \nabla f(x) = 0, s \geq 0\},$$

where $A \in \Re^{m\times n}$, $rank(A) = m$, $b \in \Re^m$ and $f : \Re^n \to R$ is a convex and twice continuously differentiable function. Suppose that the interior point condition (IPC) holds. Thus, there exist (x^0, y^0, s^0) such that

$$\text{(IPC)} \qquad \begin{aligned} Ax^0 &= b, & x^0 &> 0,\\ A^T y^0 + s^0 - \nabla f(x^0) &= 0, & s^0 &> 0. \end{aligned}$$

The IPC can be assumed without loss of generality, and we may assume $x^0 = s^0 = e$. The optimality condition for the pair (P)-(D) can be written as

$$(1) \qquad \begin{aligned} Ax &= b, & x &\geq 0,\\ A^T y + s - \nabla f(x) &= 0, & s &\geq 0,\\ xs &= 0. \end{aligned}$$

If the IPC holds, then for a fixed $\mu > 0$ the system

$$(2) \qquad \begin{aligned} Ax &= b, & x &> 0,\\ A^T y + s - \nabla f(x) &= 0, & s &> 0,\\ xs &= \mu e, \end{aligned}$$

has a unique solution, called the *μ-center*. Let us consider the function φ such that

$$(3) \qquad \varphi \in C^1,\ \varphi : \Re^+ \to \Re^+, \text{ and } \varphi^{-1} \text{ exists.}$$

Then, the system (2) is equivalent to

$$(4) \qquad \begin{aligned} Ax &= b, & x &> 0,\\ A^T y + s - \nabla f(x) &= 0, & s &> 0,\\ \varphi\left(\frac{xs}{\mu}\right) &= \varphi(e), \end{aligned}$$

Assume that we have $Ax = b$, $x > 0$, $A^T y + s - \nabla f(x) = 0$, $s > 0$ for a triple (x, y, s). Applying Newton's method for system (4) we get

$$(5) \qquad \begin{aligned} A\Delta x &= 0,\\ A^T\Delta y + \Delta s - \nabla^2 f(x)\Delta x &= 0,\\ \frac{s}{\mu}\varphi\prime\left(\frac{xs}{\mu}\right)\Delta x + \frac{x}{\mu}\varphi\prime\left(\frac{xs}{\mu}\right)\Delta s &= \varphi(e) - \varphi\left(\frac{xs}{\mu}\right). \end{aligned}$$

Denote
$$d_x = \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}.$$

We have

$$(6) \qquad \mu v(d_x + d_s) = s\Delta x + x\Delta s,$$

$$d_x d_s = \frac{\Delta x \Delta s}{\mu}. \tag{7}$$

The linear system (5) is equivalent to

$$\begin{aligned} \bar{A} d_x &= 0, \\ \bar{A}^T d_y + d_s - \bar{H} d_x &= 0, \\ d_x + d_s &= p_v, \end{aligned} \tag{8}$$

where $V = diag(v)$, $X = diag(x)$, $S = diag(s)$ and we also use the following notations:

$$p_v = \frac{\varphi(e) - \varphi(v^2)}{v\varphi\prime(v^2)}, \quad \bar{H} = \mu V S^{-1} \nabla^2 f(x) V S^{-1} \quad \text{and} \quad \bar{A} = \frac{1}{\mu} A V^{-1} X.$$

Observe that $\varphi(t) = t$ yields $p_v = v^{-1} - v$, and we obtain the standard primal-dual algorithm. Let $\varphi(t) = \sqrt{t}$. Then we have

$$p_v = 2(e - v). \tag{9}$$

From (5) we obtain

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s - \nabla^2 f(x) \Delta x &= 0, \\ \sqrt{\frac{s}{x}} \Delta x + \sqrt{\frac{x}{s}} \Delta s &= 2(\sqrt{\mu} e - \sqrt{xs}). \end{aligned} \tag{10}$$

The system (10) can be written in the following form:

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s - \nabla^2 f(x) \Delta x &= 0, \\ s\Delta x + x\Delta s &= 2(\sqrt{\mu x s} - xs). \end{aligned} \tag{11}$$

We define a proximity measure to the central path

$$\sigma(xs, \mu) = \frac{\|p_v\|}{2} = \|e - v\| = \left\| e - \sqrt{\frac{xs}{\mu}} \right\|. \tag{12}$$

Denote $q_v = d_x - d_s$. The function f is convex, thus the matrices $\nabla^2 f(x)$ and $\bar{H}$ are symmetric and positive semidefinite. Thus $d_x^T d_s \geq 0$ and

$$\|q_v\| \leq \|p_v\|.$$

We have

$$\sigma(xs, \mu) \geq \frac{\|q_v\|}{2},$$

$$d_x = \frac{p_v + q_v}{2}, \qquad d_s = \frac{p_v - q_v}{2},$$

$$d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{13}$$

Algorithm 2.1 *Let $\epsilon > 0$ be the accuracy parameter, $0 < \theta < 1$ the update parameter (default $\theta = \frac{1}{2\sqrt{n}}$), and $0 < \tau < 1$ the proximity parameter (default $\tau = \frac{1}{2}$). Suppose that for the triple (x^0, y^0, s^0) the IPC holds, and let $\mu^0 = \frac{(x^0)^T s^0}{n}$. Furthermore, suppose $\sigma(x^0 s^0, \mu^0) < \tau$.*

begin

$x := x^0;\ y = y^0;\ s = s^0;\ \mu := \mu^0;$

while $x^T s > \epsilon$ **do begin**

$\mu := (1 - \theta)\mu;$

Compute $(\Delta x, \Delta y, \Delta s)$ *from (11)*

$x := x + \Delta x;$

$y := y + \Delta y;$

$s := s + \Delta s;$

end

end.

The complexity analysis of this algorithm has been given in [17]. The results are similar to the LO analogue [5]. We revisit the following lemmas given in [17].

Lemma 2.1 *Let $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$. Moreover, let $\sigma = \sigma(xs, \mu)$ and suppose that $\sigma < 1$. Then*

$$x_+ > 0 \quad \text{and} \quad s_+ > 0,$$

hence the full Newton step is strictly feasible.

Proof: Denote $x_+(\alpha) = x + \alpha\Delta x$ and $s_+(\alpha) = s + \alpha\Delta s$ for each $0 \le \alpha \le 1$. We have

$$\frac{1}{\mu} x_+(\alpha) s_+(\alpha) = (1 - \alpha) v^2 + \alpha \left(e - (1 - \alpha) \frac{p_v^2}{4} - \alpha \frac{q_v^2}{4} \right). \tag{14}$$

$$\left\| (1 - \alpha) \frac{p_v^2}{4} + \alpha \frac{q_v^2}{4} \right\|_\infty \le \sigma^2 < 1.$$

Thus, for each $0 \le \alpha \le 1$ we have $x_+(\alpha) s_+(\alpha) > 0$. ■

Lemma 2.2 *Let $\sigma = \sigma(xs, \mu) < 1$. Then*

$$\sigma(x_+ s_+, \mu) \le 1 - \sqrt{1 - \sigma^2}.$$

Thus the Newton process is quadratically convergent.

Proof: Let $v_+ = \sqrt{\frac{x_+ s_+}{\mu}}$. Using (14) we obtain

$$v_+^2 = e - \frac{q_v^2}{4} \tag{15}$$

$$\text{(16)} \qquad \min(v_+) = \sqrt{1 - \frac{1}{4}\|q_v^2\|_\infty} \geq \sqrt{1 - \frac{\|q_v\|^2}{4}} \geq \sqrt{1-\sigma^2}.$$

$$\sigma(x_+s_+, \mu) = \left\|\frac{e - v_+^2}{e + v_+}\right\| \leq \frac{\sigma^2}{1 + \sqrt{1-\sigma^2}} = 1 - \sqrt{1-\sigma^2}.$$

Hence $\sigma(x_+s_+, \mu) < \sigma^2$, and this proves the lemma. ■

Lemma 2.3 *Let $\sigma = \sigma(xs, \mu)$ and suppose that the vectors x_+ and s_+ are obtained after a full Newton step, thus $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$. We have*

$$(x_+)^T s_+ = \mu(n - \frac{\|q_v\|^2}{4})$$

Hence $(x_+)^T s_+ \leq \mu n$.

Proof: We have

$$\frac{1}{\mu} x_+ s_+ = e - \frac{q_v^2}{4}.$$

Consequently

$$(x_+)^T s_+ = e^T(x_+s_+) = \mu(e^T e - \frac{e^T q_v^2}{4}) = \mu(n - \frac{\|q_v\|^2}{4}) = \mu(n - \sigma^2).$$

This proves the lemma. ■

Lemma 2.4 *Let $\sigma = \sigma(xs, \mu) < 1$ and $\mu_+ = (1-\theta)\mu$, where $0 < \theta < 1$. Then*

$$\sigma(x_+s_+, \mu_+) \leq \frac{\theta\sqrt{n} + \sigma^2}{1 - \theta + \sqrt{(1-\theta)(1-\sigma^2)}}$$

Moreover, if $\sigma \leq \frac{1}{2}$, $\theta = \frac{1}{2\sqrt{n}}$ and $n \geq 4$ then we have $\sigma(x_+s_+, \mu_+) \leq \frac{1}{2}$.

Proof: Using (15) and (16) we may write

$$\sigma(x_+s_+, \mu_+) = \left\|e - \sqrt{\frac{x_+s_+}{\mu_+}}\right\| = \frac{1}{\sqrt{1-\theta}} \left\|\frac{(1-\theta)e - v_+^2}{\sqrt{1-\theta}e + v_+}\right\| \leq$$

$$\leq \frac{1}{1 - \theta + \sqrt{(1-\theta)(1-\sigma^2)}} \left\|-\theta e + \frac{q_v^2}{4}\right\| \leq \frac{\theta\sqrt{n} + \sigma^2}{1 - \theta + \sqrt{(1-\theta)(1-\sigma^2)}}$$

This implies the first part of the lemma. To prove the second part observe that for $n \geq 4$ and $\theta = \frac{1}{2\sqrt{n}}$ we have $1 - \theta \geq \frac{3}{4}$. Finally, for $\sigma \leq \frac{1}{2}$ a simple calculus yields $\sigma(x_+s_+, \mu_+) \leq \frac{1}{2}$. ■

Lemma 2.5 *Suppose that $x^0 = s^0 = e$. Then Algorithm 2.1 performs at most*

$$\left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil$$

interior point iterations.

Proof: For the proof we refer to [17], and for the LO variant [5]. ■

Thus, we obtain the following theorem.

Theorem 2.6 *Suppose that $x^0 = s^0 = e$. Using the default values for θ and τ Algorithm 2.1 requires no more than*

$$\left\lceil 2\sqrt{n}\log\frac{n}{\epsilon}\right\rceil$$

interior point iterations. The resulting vectors satisfy $x^T s \le \epsilon$. ■

3. Predictor-corrector algorithm

The third equation in the system (8) can be written in the form:

$$d_x + d_s = 2e - 2v.$$

Observe that the expression on the right hand side can be viewed as a sum of two terms. Consider the following equations.

$$d_x^a + d_s^a = -2v, \tag{17}$$

$$d_x^c + d_s^c = 2e, \tag{18}$$

and conclude that the standard Newton direction has been breaking down into two steps, the affine-scaling, or predictor one: d_x^a and d_s^a, and the centering, or corrector step: d_x^c and d_s^c. The equations (17) and (18) yield the following systems:

$$\begin{aligned} \bar{A}d_x^a &= 0,\\ \bar{A}^T d_y^a + d_s^a - \bar{H}d_x^a &= 0,\\ d_x^a + d_s^a &= -2v, \end{aligned} \tag{19}$$

and

$$\begin{aligned} \bar{A}d_x^c &= 0,\\ \bar{A}^T d_y^c + d_s^c - \bar{H}d_x^c &= 0,\\ d_x^c + d_s^c &= 2e, \end{aligned} \tag{20}$$

where $\bar{A} = \frac{1}{\mu}Adiag(\frac{x}{v})$ and $\bar{H} = \mu VS^{-1}\nabla^2 f(x)VS^{-1}$. The systems (19) and (20) have unique solutions. Denote by d_x^a, d_s^a, d_x^c and d_s^c these solutions. We have

$$(d_x^a)^T d_s^a \ge 0, \qquad (d_x^c)^T d_s^c \ge 0, \tag{21}$$

and the solution of (8) can be obtained from (19) and (20) as follows

$$d_x = d_x^a + d_x^c,$$

$$d_s = d_s^a + d_s^c.$$

The step direction vectors in the original space are

$$\Delta^a x = \frac{x}{v} d_x^a, \qquad \Delta^a s = \frac{s}{v} d_s^a, \qquad \Delta^a y = d_y^a,$$

$$\Delta^c x = \frac{x}{v} d_x^c, \qquad \Delta^c s = \frac{s}{v} d_s^c, \qquad \Delta^c y = d_y^c.$$

Thus

$$x\Delta^a s + s\Delta^a x = \mu v(d_x^a + d_s^a) = -2\mu v^2 = -2xs, \tag{22}$$

$$x\Delta^c s + s\Delta^c x = \mu v(d_x^c + d_s^c) = 2\mu v = 2\sqrt{xs\mu}, \tag{23}$$

$$\Delta^a x \Delta^a s = \mu d_x^a d_s^a. \tag{24}$$

From (22) we obtain that $(\Delta^a x, \Delta^a y, \Delta^a s)$ is the solution of the system:

$$\begin{aligned} A\Delta^a x &= 0, \\ A^T\Delta^a y + \Delta^a s - \nabla^2 f(x)\Delta^a x &= 0, \\ s\Delta^a x + x\Delta^a s &= -2xs. \end{aligned} \tag{25}$$

Now we are ready to describe the predictor-corrector algorithm.

Algorithm 3.1

Let $0 < \tau < 1$ be the proximity parameter (default value $\tau = \frac{5}{13}$), $\epsilon > 0$ the accuracy parameter, and $0 < \theta < \frac{1}{2}$ the update parameter (default $\theta = \frac{1}{3\sqrt{n}}$). Assume that for the triple (x^0, y^0, s^0) IPC holds, and let $\mu^0 = \frac{(x^0)^T s^0}{n}$. Furthermore, assume that $\sigma(x^0 s^0, \mu^0) \le \tau$.

begin
  $x := x^0$; $s := s^0$; $\mu := \mu^0$;
  while $x^T s > \epsilon$ **do begin**
    Compute $(\Delta x, \Delta y, \Delta s)$ from (11).
    $x := x + \Delta x$;
    $s := s + \Delta s$;
    Compute $(\Delta^a x, \Delta^a y, \Delta^a s)$ using the system (25).
    $x := x + \theta\Delta^a x$;
    $s := s + \theta\Delta^a s$;
    $\mu := (1 - 2\theta)\mu$;
  end
end.

Our first aim is to prove that this algorithm is well defined. We discuss also the complexity of the algorithm. To achieve these goals, in the first lemma we find lower and upper bounds for the components of the vector v.

Lemma 3.1 *Let $x > 0$, $s > 0$, $\mu > 0$, $v = \sqrt{\frac{xs}{\mu}}$ and $\sigma = \sigma(xs, \mu) = \|e - v\|$. Assume that $\sigma < 1$. Then, for all i such that $1 \le i \le n$, we have*

$$1 - \sigma \le v_i \le 1 + \sigma.$$

Moreover, the following inequalities hold

$$\min\left(v^2\right) \geq (1-\sigma)^2, \quad \|v\|^2 \leq n(1+\sigma)^2. \tag{26}$$

Proof: See [6]. ■

The following lemma provides upper bounds for the Euclidian norm and the infinity norm of the product of two vectors with non-negative inner product.

Lemma 3.2 (Extension of the first uv-lemma in [11]**)** *Let $\pi \in \Re^n$ and $\zeta \in \Re^n$ are two vectors such that $\pi^T\zeta \geq 0$. Then we have*

$$\|\pi\zeta\|_\infty \leq \frac{1}{4}\|\pi+\zeta\|^2, \quad \|\pi\zeta\| \leq \frac{\sqrt{2}}{4}\|\pi+\zeta\|^2.$$

Proof:

We have $\|\pi+\zeta\|^2 = \|\pi-\zeta\|^2 + 4\pi^T\zeta$, resulting $\|\pi+\zeta\| \geq \|\pi-\zeta\|$. Furthermore

$$\pi\zeta = \frac{1}{4}(\pi+\zeta)^2 - \frac{1}{4}(\pi-\zeta)^2. \tag{27}$$

Thus

$$-\frac{1}{4}(\pi-\zeta)^2 \leq \pi\zeta \leq \frac{1}{4}(\pi+\zeta)^2,$$

and

$$-\frac{1}{4}\|\pi+\zeta\|^2 e \leq -\frac{1}{4}\|\pi-\zeta\|^2 e \leq \pi\zeta \leq \frac{1}{4}\|\pi+\zeta\|^2 e.$$

This proves the first inequality. To prove the second one, observe that from (27) and $\|\pi\zeta\|^2 = e^T(\pi\zeta)^2$ we obtain:

$$\|\pi\zeta\|^2 = \frac{1}{16}e^T\left((\pi+\zeta)^2 - (\pi-\zeta)^2\right)^2 \leq \frac{1}{16}e^T\left((\pi+\zeta)^4 + (\pi-\zeta)^4\right).$$

Moreover, for each $\xi \in \Re^n$ the $e^T\xi^4 \leq \|\xi\|^4$ inequality holds, therefore

$$\|\pi\zeta\|^2 \leq \frac{1}{16}\|\pi+\zeta\|^4 + \frac{1}{16}\|\pi-\zeta\|^4,$$

and using again the relation $\|\pi-\zeta\| \leq \|\pi+\zeta\|$ we get the second inequality. This proves the lemma. ■

We introduce the followig notations. Let

$$\lambda(\sigma) = \left(1+\sqrt{2}\right)\sigma^2 - 2\left(\sqrt{2}-1\right)\sigma + \sqrt{2}, \tag{28}$$

$$K(\sigma,\theta,n) = (1-\sigma)^2 - \frac{\theta^2 n}{1-2\theta}\left(1+\sigma\right)^2, \tag{29}$$

and

$$\Phi(\sigma,\theta,n) = \frac{\lambda(\sigma) - \sqrt{2}K(\sigma,\theta,n)}{1+\sqrt{K(\sigma,\theta,n)}}. \tag{30}$$

Consider the following function

$$\omega(t) = 1 - \sqrt{1-t^2}, \tag{31}$$

defined for every $0 \le t < 1$. For fixed τ introduce the function

$$\Psi_\tau(\sigma) = \frac{1}{(1+\sigma)^2}\left(\sigma^2 - 2\sigma + \sqrt{2}\tau - \frac{\tau^2}{2}\right). \tag{32}$$

We give a sufficient condition for yielding strictly feasible vectors after an affine-scaling step.

Lemma 3.3 *Let $x > 0$, $s > 0$, $\mu > 0$ in such a way, that $\sigma = \sigma(xs, \mu) < 1$. Furthermore, let $0 < \theta < \frac{1}{2}$. Denote $x^+ = x + \theta\Delta^a x$ and $s^+ = s + \theta\Delta^a s$. Then*

$$x^+ > 0 \quad \text{and} \quad s^+ > 0,$$

if the inequality $K(\sigma, \theta, n) > 0$ holds.

Proof: Let us introduce the notations

$$x^+(\beta) = x + \beta\theta\Delta^a x \quad \text{and} \quad s^+(\beta) = s + \beta\theta\Delta^a s$$

for each real number $0 \le \beta \le 1$. We have the following equality:

$$x^+(\beta)s^+(\beta) = xs + \beta\theta(x\Delta^a s + s\Delta^a x) + \beta^2\theta^2\Delta^a x\Delta^a s.$$

Using the relations (22) and (24) we get:

$$x^+(\beta)s^+(\beta) = (1-2\beta\theta)xs + \mu\beta^2\theta^2 d_x^a d_s^a \tag{33}$$

thus we obtain

$$\frac{x^+(\beta)s^+(\beta)}{(1-2\beta\theta)\mu} = v^2 + \frac{\beta^2\theta^2}{(1-2\beta\theta)} d_x^a d_s^a. \tag{34}$$

Therefore

$$\min\left(\frac{x^+(\beta)s^+(\beta)}{(1-2\beta\theta)\mu}\right) \ge \min\left(v^2\right) - \frac{\beta^2\theta^2}{1-2\beta\theta}\left\|d_x^a d_s^a\right\|_\infty.$$

Moreover, for each fixed $0 < \theta < \frac{1}{2}$, the function $\vartheta(\beta) = \frac{\beta^2\theta^2}{1-2\beta\theta}$ defined for $0 \le \beta \le 1$ is strictly increasing, thus

$$\min\left(\frac{x^+(\beta)s^+(\beta)}{(1-2\beta\theta)\mu}\right) \ge \min\left(v^2\right) - \frac{\theta^2}{1-2\theta}\left\|d_x^a d_s^a\right\|_\infty. \tag{35}$$

From Lemma 3.2, using the equality (17) and Lemma 3.1 we obtain

$$\left\|d_x^a d_s^a\right\|_\infty \le \frac{1}{4}\left\|d_x^a + d_s^a\right\|^2 = \|v\|^2 \le n(1+\sigma)^2. \tag{36}$$

Now, using the relation (35) and Lemma 3.1 again, we get

$$\min\left(\frac{x^+(\beta)s^+(\beta)}{(1-2\beta\theta)\mu}\right) \ge K(\sigma, \theta, n) \tag{37}$$

But $K(\sigma,\theta,n) > 0$, and we deduce that for each $0 \le \beta \le 1$ the inequality $x^+(\beta)s^+(\beta) > 0$ holds. Therefore the $x^+(\beta)$ and $s^+(\beta)$ functions are not changing sign on the $[0,1]$ interval. We know that $x^+(0) = x > 0$, and $s^+(0) = s > 0$, thus we conclude that $x^+(1) = x^+ > 0$, and $s^+(1) = s^+ > 0$. This proves the lemma. ∎

In the next lemma we investigate the modification of the proximity measure after an affine-scaling step, and the update of the parameter μ.

Lemma 3.4 *Let $x > 0$, $s > 0$, $\mu > 0$ such that $\sigma = \sigma(xs,\mu) < 1$. Moreover, let $0 < \theta < \frac{1}{2}$ and assume that $K(\sigma,\theta,n) > 0$. Assume that we obtain the vectors x^+ and s^+ from an affine-scaling step, thus $x^+ = x + \theta\Delta^a x$ and $s^+ = s + \theta\Delta^a s$. Denote $\mu^+ = (1-2\theta)\mu$ and $\sigma^+ = \sigma(x^+s^+,\mu^+)$. Then the inequality*

$$\sigma^+ \le \Phi(\sigma,\theta,n) \tag{38}$$

holds.

Proof: From Lemma 3.3 we deduce that the affine-scaling step is strictly feasible. Denote

$$v^+ = \sqrt{\frac{x^+s^+}{\mu^+}}.$$

By substituting $\beta = 1$ in the relations (34) and (37) we get

$$\left(v^+\right)^2 = v^2 + \frac{\theta^2}{1-2\theta} d_x^a d_s^a, \tag{39}$$

$$\min\left(v^+\right) \ge \sqrt{K(\sigma,\theta,n)}. \tag{40}$$

Moreover

$$\sigma^+ = \left\|e - v^+\right\| = \left\|\frac{e - \left(v^+\right)^2}{e + v^+}\right\|,$$

so the following inequality holds

$$\sigma^+ \le \frac{\left\|e - v^2\right\| + \left\|v^2 - \left(v^+\right)^2\right\|}{1 + \min\left(v^+\right)}. \tag{41}$$

Using Lemma 3.2, the equality (17) and Lemma 3.1 we obtain

$$\|d_x^a d_s^a\| \le \frac{\sqrt{2}}{4}\,\|d_x^a + d_s^a\|^2 = \sqrt{2}\|v\|^2 \le \sqrt{2}n(1+\sigma)^2. \tag{42}$$

Now, from (39) we get

$$\left\|v^2 - \left(v^+\right)^2\right\| \le \frac{\theta^2}{1-2\theta}\sqrt{2}n(1+\sigma)^2. \tag{43}$$

Observe that $\left\|e - v^2\right\| \leq \sigma + \|v(e-v)\|$, and Lemma 3.1 yields $\|v\|_\infty \leq 1 + \sigma$, therefore

$$\left\|e - v^2\right\| \leq \sigma + \|v\|_\infty \|e - v\| \leq \sigma^2 + 2\sigma. \tag{44}$$

Finally, using the relations (40), (41), (43) and (44) we deduce

$$\sigma^+ \leq \frac{\sigma^2 + 2\sigma + \frac{\theta^2}{1-2\theta}\sqrt{2}n(1+\sigma)^2}{1 + \sqrt{K(\sigma,\theta,n)}} \tag{45}$$

and this results in (38). Thus, the lemma is proved. ■

The next lemma is devoted to the proximity measure of the vectors obtained by a full Newton step. We use also the results of Lemma 2.2.

Lemma 3.5 *Let $x > 0$, $s > 0$, $\mu > 0$, and $0 < \tau < 1$ in such a way that $\sigma = \sigma(xs, \mu) \leq \tau$. Suppose that the vectors x^+ and s^+ are produced by a full Newton process, thus $x^+ = x + \Delta x$ and $s^+ = s + \Delta s$. Then*

$$\sigma(x^+ s^+, \mu) \leq \omega(\tau). \tag{46}$$

Moreover, if $\tau \leq \frac{3}{4}$, then $\sigma(x^+s^+, \mu) < 6 - 4\sqrt{2}$ and $\sigma(x^+s^+, \mu) < \frac{\tau}{\sqrt{2}}$.

Proof: From Lemma 2.2 we obtain

$$\sigma(x^+s^+, \mu) \leq \omega(\sigma).$$

Furthermore, the function $\omega(t)$ is increasing for $0 \leq t < 1$, so the inequality (46) holds. If we assume $\tau \leq \frac{3}{4}$, then a simple calculus yields

$$\sigma(x^+s^+, \mu) \leq \omega(\tau) \leq \omega\left(\frac{3}{4}\right) = 1 - \frac{\sqrt{7}}{4} < 6 - 4\sqrt{2}.$$

For the last relation it is sufficient to prove that the inequality $\omega(\tau) < \frac{\tau}{\sqrt{2}}$ holds. We have $\tau \leq \frac{3}{4} < \frac{2\sqrt{2}}{3}$, thus $3\tau^2 < 2\sqrt{2}\tau$, therefore $\left(\sqrt{2}\tau - 1\right)^2 = 2\tau^2 - 2\sqrt{2}\tau + 1 < 1 - \tau^2$. We obtain

$$\frac{\tau}{1 + \sqrt{1 - \tau^2}} < \frac{1}{\sqrt{2}},$$

and from this inequality we get $\omega(\tau) < \frac{\tau}{\sqrt{2}}$. This proves the lemma. ■

In the following lemma we provide a sufficient condition, which guarantees that after an affine-scaling step the proximity measure will not exceed the proximity parameter.

Lemma 3.6 *Let τ be fixed such that $0 < \tau \leq \frac{3}{4}$ and let $\mu > 0$. Assume now that $x > 0$ and $s > 0$ are the vectors generated by the full Newton step of Algorithm 3.1. Let x^+ and s^+ be the vectors obtained after the affine-scaling step, thus $x^+ = x + \theta\Delta^a x$ and $s^+ = s + \theta\Delta^a s$. Denote $\mu^+ = (1 - 2\theta)\mu$*

and $\sigma^+ = \sigma(x^+ s^+, \mu^+)$*. Finally, assume that the update parameter satisfies the* $0 < \theta < \frac{1}{2}$ *condition. Then the inequality*

$$\sigma^+ \leq \tau \tag{47}$$

holds if

$$\frac{\theta^2 n}{1 - 2\theta} \leq \Psi_\tau(\sigma), \tag{48}$$

Moreover, $\Psi_\tau(\sigma) > 0$ *and for each fixed* τ *the function* Ψ_τ *is decreasing on the closed interval* $[0,\ \omega(\tau)]$*.*

Proof: Let us introduce the notation

$$\psi_\tau(\sigma) = \sigma^2 - 2\sigma + \sqrt{2}\tau - \frac{\tau^2}{2}. \tag{49}$$

Thus

$$\Psi_\tau(\sigma) = \frac{\psi_\tau(\sigma)}{(1+\sigma)^2}.$$

Observe that for $0 < \tau \leq \frac{3}{4}$ we have $1 - \frac{\tau}{\sqrt{2}} \geq 1 - \frac{3\sqrt{2}}{8} > 0$. Therefore

$$\psi_\tau(\sigma) = (1-\sigma)^2 - \left(1 - \frac{\tau}{\sqrt{2}}\right)^2 < (1-\sigma)^2 \tag{50}$$

Suppose that the inequality (48) holds. Then, from (50) results $K(\sigma, \theta, n) > 0$, and we obtain that Lemma 3.4 can be applied. Because $x > 0$ and $s > 0$ are the vectors generated by a full Newton step of Algorithm 3.1, we deduce that there exists the vectors $\tilde{x} > 0$ and $\tilde{s} > 0$ in such a way that, from these vectors we obtain x and s by a full Newton step. Moreover, $\sigma(\tilde{x}\tilde{s}, \mu) \leq \tau$, and applying Lemma 3.5 for the vectors $\tilde{x}$ and $\tilde{s}$ we get the following inequalities

$$\sigma < \frac{\tau}{\sqrt{2}}, \tag{51}$$

$$\sigma < 6 - 4\sqrt{2}. \tag{52}$$

Lemma 3.4 implies that the inequality $\sigma^+ \leq \tau$ holds if $\Phi(\sigma, \theta, n) \leq \tau$. This can be written in the following form

$$\sqrt{2}K(\sigma, \theta, n) + \tau\sqrt{K(\sigma, \theta, n)} + \tau - \lambda(\sigma) \geq 0.$$

Denote $\kappa = \sqrt{K(\sigma, \theta, n)}$ and $\varrho(t) = \sqrt{2}t^2 + \tau t + \tau - \lambda(\sigma)$. Then (47) holds if $\varrho(\kappa) \geq 0$. The next issue is to determine lower and upper bounds of $\lambda(\sigma)$ and using these results to study the sign of the function ϱ. From (52) we get $0 \leq \sigma < 6 - 4\sqrt{2}$, therefore

$$\left(1 + \sqrt{2}\right)\sigma^2 \leq 2\left(\sqrt{2} - 1\right)\sigma,$$

thus $\lambda(\sigma) \le \sqrt{2}$. Moreover, we have

$$\lambda(\sigma) = \left(1+\sqrt{2}\right)\left(\sigma - \left(3-2\sqrt{2}\right)\right)^2 + 7 - 4\sqrt{2} \ \ge\ 7 - 4\sqrt{2},$$

resulting in

$$7 - 4\sqrt{2} \le \lambda(\sigma) \le \sqrt{2}. \tag{53}$$

Let

$$\Delta_{\sigma,\tau} = \tau^2 - 4\sqrt{2}\,\tau + 4\sqrt{2}\,\lambda(\sigma). \tag{54}$$

The roots of the $\varrho(t) = 0$ equation are

$$t_1 = \frac{-\tau - \sqrt{\Delta_{\sigma,\tau}}}{2\sqrt{2}}, \quad t_2 = \frac{-\tau + \sqrt{\Delta_{\sigma,\tau}}}{2\sqrt{2}}.$$

Since $0 < \tau \le \frac{3}{4}$, from (53) we get $\tau < \lambda(\sigma)$ thus

$$\Delta_{\sigma,\tau} > \tau^2 > 0.$$

We obtain the inequalities $t_1 < 0$ and $t_2 > 0$, and this means that if $\kappa \ge t_2$, then $\varrho(\kappa) \ge 0$ holds, and (47) is satisfied. Using (53) from (54) we get

$$\sqrt{\Delta_{\sigma,\tau}} \le \sqrt{\tau^2 - 4\sqrt{2}\,\tau + 8} = \sqrt{\left(2\sqrt{2} - \tau\right)^2} = 2\sqrt{2} - \tau,$$

therefore

$$t_2 \le \frac{2\sqrt{2} - 2\tau}{2\sqrt{2}} = 1 - \frac{\tau}{\sqrt{2}}.$$

We deduce that the inequality (47) holds if $\kappa \ge 1 - \frac{\tau}{\sqrt{2}}$, and this can be written in the form

$$K(\sigma, \theta, n) \ge \left(1 - \frac{\tau}{\sqrt{2}}\right)^2.$$

Using (29) the inequality (48) follows. This proves the first assertion of the lemma. Now, from Lemma 3.5 the inequality $\sigma \le \omega(\tau)$ holds, therefore we are going to study the functions ψ_τ and Ψ_τ on the $[0,\ \omega(\tau)]$ interval. Observe, that from (50) we have

$$\psi_\tau(\sigma) = \left(\frac{\tau}{\sqrt{2}} - \sigma\right)\left(2 - \frac{\tau}{\sqrt{2}} - \sigma\right).$$

Since

$$2 - \frac{\tau}{\sqrt{2}} - \sigma \ >\ 2\left(1 - \frac{\tau}{\sqrt{2}}\right) > 0$$

we obtain the inequality $\psi_\tau(\sigma) > 0$, which results in $\Psi_\tau(\sigma) > 0$. Taking the derivative of the function ψ_τ we get

$$\left(\psi_\tau\right)'(\sigma) = 2\sigma - 2 < 0.$$

Thus, the function ψ_τ is positive and is decreasing on the interval $[0,\ \omega(\tau)]$. The same is true for the function $\frac{1}{(1+\sigma)^2}$ and this implies the last result of the lemma. ■

In Lemma 3.7 we investigate how will be modified the duality gap after an affine-scaling step.

Lemma 3.7 *Let $x > 0$, $s > 0$ and $\mu > 0$ such that $\sigma = \sigma(xs, \mu) < 1$ and $0 < \theta < \frac{1}{2}$. Assume that x^+ and s^+ are the vectors obtained after the affine-scaling step of Algorithm 3.1. Then the following inequality holds*

$$\left(x^+\right)^T s^+ \leq (1-2\theta+2\theta^2)x^T s \leq (1-\theta)x^T s.$$

Proof: Substitute $\beta = 1$ in the relation (33). Thus

$$\left(x^+\right)^T s^+ = e^T\left(x^+ s^+\right) = (1-2\theta)e^T(xs) + \mu\theta^2 e^T\left(d_x^a d_s^a\right).$$

Since (17) we have

$$d_x^a d_s^a = 2v^2 - \frac{(d_x^a)^2 + (d_s^a)^2}{2}$$

and this leads to

$$e^T(d_x^a d_s^a) = 2e^T\frac{xs}{\mu} - \frac{\|d_x^a\|^2 + \|d_s^a\|^2}{2} \leq \frac{2}{\mu}x^T s.$$

We obtain $\mu\theta^2 e^T\left(d_x^a d_s^a\right) \leq 2\theta^2 x^T s$ and this implies the first inequality of the lemma. Now observe that for $0 < \theta < \frac{1}{2}$ the inequality

$$1 - 2\theta + 2\theta^2 \leq 1 - \theta$$

holds, thus we get the second inequality. This proves the lemma. ■

Lemma 3.8 is devoted to finding an upper bound for the duality gap after a whole iteration (full Newton step followed by an affine-scaling step).

Lemma 3.8 *Let $x > 0$, $s > 0$ and $\mu > 0$ such that $\sigma = \sigma(xs, \mu) < 1$ and $0 < \theta < \frac{1}{2}$. Assume that the vectors x^+ and s^+ are obtained after an iteration of Algorithm 3.1. Moreover, let $\mu^+ = (1-2\theta)\mu$. Then the relation*

$$\left(x^+\right)^T s^+ \leq (1-\theta)n\mu < \frac{n\mu^+}{1-2\theta}$$

is satisfied.

Proof: Let $\bar{x}$ and $\bar{s}$ be the vectors obtained by a full Newton step. Using Lemma 2.3 we get $\bar{x}^T\bar{s} \leq n\mu$. Furthermore, from Lemma 3.7 and $1-\theta < 1$ we obtain

$$\left(x^+\right)^T s^+ = (1-\theta)\bar{x}^T\bar{s} \leq (1-\theta)n\mu < \frac{n\mu^+}{1-2\theta}.$$

This completes the proof. ■

In the following lemma we analyse the question of the bound on the number of iterations performed by the algorithm. We assume that we would like to approximate the optimal solution with a given precision.

Lemma 3.9 *Let x^k and s^k be the vectors generated by Algorithm 3.1 after k iterations, where $k > 1$. Then for each k satisfying the condition*

$$k \geq 1 + \left\lceil \frac{1}{2\theta} \log \frac{(x^0)^T s^0}{\epsilon} \right\rceil$$

the inequality $(x^k)^T s^k < \epsilon$ holds.

Proof: Let μ^k be the value of μ after k iterations. From Lemma 3.8 results

$$(x^k)^T s^k < \frac{n\mu^k}{1-2\theta} = (1-2\theta)^{k-1} n\mu^0 = (1-2\theta)^{k-1}(x^0)^T s^0.$$

Thus the inequality $(x^k)^T s^k < \epsilon$ holds if

$$(1-2\theta)^{k-1}(x^0)^T s^0 \leq \epsilon.$$

Taking logarithms we obtain

$$(k-1)\log(1-2\theta) + \log((x^0)^T s^0) \leq \log \epsilon$$

and using the relation $-\log(1-2\theta) \geq 2\theta$ we conclude that this inequality is satisfied if

$$2\theta(k-1) \geq \log((x^0)^T s^0) - \log \epsilon = \log \frac{(x^0)^T s^0}{\epsilon}.$$

This implies the lemma. ■

In the following theorem we give a sufficient condition, which guarantees that the algorithm will be well defined. Furthermore, we provide an upper bound for the number of iterations.

Theorem 3.10 *Let $0 < \tau \leq \frac{3}{4}$ and $0 < \theta < \frac{1}{2}$. If*

$$\frac{\theta^2 n}{1-2\theta} \leq \Psi_\tau(\omega(\tau)), \tag{55}$$

then Algorithm 3.1 is well defined and performs at most

$$1 + \left\lceil \frac{1}{2\theta} \log \frac{(x^0)^T s^0}{\epsilon} \right\rceil \tag{56}$$

iterations. The generated vectors satisfy the $x^T s < \epsilon$ inequality.

Proof: As in the proof of Lemma 3.6 let $\tilde{x}$ and $\tilde{s}$ be the vectors at the begining of a new iterate. Furthermore, let x and s be the vectors after the full Newton step. Finaly, denote by x^+ and s^+ the vectors obtained by the affine-scaling step. We have to prove that the interior point condition holds every time a new iterate begins and the proximity measure is not greater than τ. This assertion will be true if we have the following one. Suppose that $\tilde{x} > 0$, $\tilde{s} > 0$ and $\sigma(\tilde{x}\tilde{s}, \mu) \leq \tau$ then we have to prove that $x^+ > 0$ and $s^+ > 0$ and for the proximity measure we have the inequality $\sigma(x^+s^+, \mu^+) \leq \tau$, where μ^+ denotes the value of the parameter μ at the end of the iteration.

From Lemma 2.1 results $x > 0$ and $s > 0$. Using these relations, from Lemma 3.3 we obtain that the inequalities $x^+ > 0$ and $s^+ > 0$ are satisfied if $K(\sigma, \theta, n) > 0$. Moreover, using Lemma 3.6 the inequality $\sigma(x^+s^+, \mu^+) \leq \tau$ holds if we have the relation (48) and from (50) we deduce that in this case the inequality $K(\sigma, \theta, n) > 0$ is also satisfied.

This means that it is sufficient to prove that the inequality (48) holds. From Lemma 3.5 we deduce

$$\sigma = \sigma(xs, \mu) \leq \omega(\tau).$$

Since the function Ψ_τ is decreasing, we conclude that the inequality (48) is satisfied if the relation (55) holds. Lemma 3.9 implies the upper bound for the number of iterations. This completes the proof. ■

In the next theorem we prove that Algorithm 3.1 is well defined for the default values. From the upper bound on the number of iterations we conclude that this predictor-corrector type algorithm finds an ϵ-solution in polynomial time.

Theorem 3.11 *Let $\tau = \frac{5}{13}$ and $\theta = \frac{1}{3\sqrt{n}}$, where $n \geq 2$. Then Algorithm 3.1 is well defined and requires no more than*

$$\left\lceil 3\sqrt{n} \log \frac{(x^0)^T s^0}{\epsilon} \right\rceil$$

iterations. For the vectors obtained we have the $x^T s \leq \epsilon$ inequality. ■

4. Conclusion

We have introduced a new predictor-corrector algorithm for solving LCCO problems. The method of finding a new search direction is based on an equivalent algebraic transformation of the centering equation from the system, which defines the central path. Polynomial complexity is proved, and the best known iteration bound for small-update methods is obtained.

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