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AUTOMATED PROOF OF GEOMETRY THEOREMS INVOLVING ORDER RELATION IN THE FRAME OF THE *THEOREMA* PROJECT

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ABSTRACT. Collins' Cylindrical Algebraic Decomposition method (CAD) can be used to prove geometry theorems that involve order relation (that is, the algebraic form consists of polynomial equalities and inequalities). Unfortunately only very simple geometric statements can be proved this way, as the method is very time consuming. To overcome the slowness of Collins' CAD method for complicated polynomials we propose a method (section 4) that combines the area method for computing geometric quantities and the CAD method. We present an implementation of this method as part of the Geometry Prover in the frame of the *Theorema* project.

1. INTRODUCTION

In this paper we present a method for proving geometry theorems that involve order relation (that is, the algebraic form consists of polynomial equalities and inequalities). We deal with a class of statements in plane Euclidean geometry, that we call constructive geometry statements possibly involving inequalities. These statements are constructive statements in the sense presented in [5], but they may also contain further constraints for the constructed points. This way we can also deal with notions like a point on a segment, incircle or interior bisector.

Collins' Cylindrical Algebraic Decomposition method (we shall refer to it as CAD method) introduced in [6], improving earlier work of [9], can be used to prove geometry theorems that involve order relation as has been observed in [7], [1] and other papers. Unfortunately only very simple geometric statements can be proved this way, as the method is very time consuming.

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To overcome the slowness of Collins' CAD method for complicated polynomials we propose a method that combines the area method presented in [5] and the CAD method. First we compute the expressions involved in inequalities using the area method. This way we obtain a new problem equivalent to the original one that is expressed only in terms of the free points (arbitrary points, introduced by a *point* construction, see Definition 1) of the original constructions. Applying the CAD method to this new problem we can obtain the result in reasonable time even for quite complicated problems.

The Geometry Prover is part of *Theorema*, a mathematical software system implemented in *Mathematica* and, hence, available on all computer platforms for which *Mathematica* is available. *Theorema* aims at providing one uniform logical and software technological frame for automated theorem proving in all areas of mathematics or, in other words and more generally, for formal mathematics, i.e. proving, solving, and simplifying mathematical formulae relative to mathematical knowledge bases, see [2], [3]. *Theorema* is being developed at the RISC Institute by the *Theorema* Group under the direction of Bruno Buchberger. For a presentation of *Theorema* compared to other existing provers see [10].

Theorema offers a user-friendly interface for problem input. It generates fully automatically the proofs that contain all the necessary explanations.

The Geometry Prover is based on the methods described in [11], [4], [7], [5]. The input for the geometry prover, i.e. the algebraic formulation of all the construction steps and of the property the final configuration should satisfy is generated automatically from the geometric description of the theorem.

In the next sections we shall

- define the object of our study: the class of constructive geometry statements possibly involving inequalities in plane Euclidean geometry;
- give a brief description of the area method and our notations relative to this method;
- present our method based on combining the area method and Collins' CAD method for proving the above class of statements;
- give an example.

2. Constructive Geometry Statements Possibly Involving Inequalities

In this paper we deal with a class of statements in plane Euclidean geometry. We consider three kinds of geometric objects: points, lines and circles. To make sure that the constructed objects are welldefined, we need to assume some nondegenerate conditions (denoted by ndg conditions in what follows). These conditions are automatically generated by the prover.

A straight line can be defined either by two distinct points, or a point and its direction. So it can be given in one of the following forms:

$line[A, B]$ is the line passing through points A and B.

$pline[C, A, B]$ is the line passing through point C, parallel to $line[A, B]$.

$tline[C, A, B]$ is the line passing through point C, perpendicular to $line[A, B]$. To make sure that all three kinds of lines are welldefined, we need to assume $A \neq B$.

$circle[O, A]$ is the circle with point O as its center and passing through point A. Again, $A \neq O$ has to be assumed.

Definition 1. *A construction is one of the following ways to introduce new points. For each construction we also give its ndg conditions.*

C1: $point[A_1, \ldots, A_n]$. *Take arbitrary points* $A_1, \ldots, A_n$ *in the plain. Each* A_i *is a free point.*

C2: $pon[A, ln]$. *Take a point* A *on line* ln. *The ndg condition of C2 is the ndg condition of line* ln.

C3: $pon[A, circle[O, U]]$. *Take a point* A *on* $circle[O, U]$. *The ndg condition is* $O \neq U$.

C4: $inter[A, ln1, ln2]$. *Let point* A *the intersection of line* $ln1$ *and line* $ln2$. *The ndg condition is* $ln1 \nparallel ln2$.

C5: $inter[A, ln, circle[O, P]]$. *Introduce point* A *as the intersection of line* ln *and* $circle[O, P]$. *The ndg condition is* $P \neq O$ *and line* ln *is not degenerate.*

C6: $inter[A, circle[O_1, P], circle[O_2, P]]$. *Point* A *is the other intersection point of* $circle[O_1, P]$ *and* $circle[O_2, P]$. *The ndg condition is* $P \neq O_1$ *and* $P \neq O_2$.

C7: $pratio[A, W, U, V, r]$. *Take a point* A *on the line passing through* W *and parallel to line* UV *such that* $\overline{WA} = r\overline{UV}$, *where* r *can be a rational number, a rational expression of some geometric quantities, or a variable. The ndg. condition is* $U \neq V$. $\overline{UV}$ *denotes the length of the oriented segment* UV.

C8: $tratio[A, U, V, r]$. *Take a point* A *on the line passing through* U *and perpendicular to line* UV *such that* $\overline{UA} = r\overline{UV}$, *where* r *can be a rational number, a rational expression of some geometric quantities, or a variable. The ndg. condition is* $U \neq V$.

The point A in each construction is said to be introduced by that construction.

Definition 2. *A constructive geometry statement possibly involving inequalities is a list* $S = (C, H, G)$, *where*

1. $C = \{C_1, C_2, \ldots, C_m\}$ *is a construction set. Each* C_i *is a construction such that the point introduced by it must be different from points introduced by* $C_j, j = 1, \ldots, i-1$ *and other points occurring in* C_i *must be introduced before;*
2. $H = \{H_1, H_2, \ldots, H_n\}$ *is a set of additional geometric properties whose algebraic representation may involve inequalities. All the points appearing in* H *have to be introduced by the constructions.*
3. $G = \{G_1, G_2, \ldots, G_k\}$, *the conclusion, is a set of geometric properties of the points introduced by the constructions (it may also contain inequalities).*

3. The Area Method

As a first step of our proof we use the area method, a coordinatefree technique for proving geometry theorems based on point elimination. The basic geometry invariants used are the signed area and Pythagoras differences of oriented triangles and ratios of oriented segments. The method can deal with geometric statements of constructive type, where each new point is introduced by one construction using only previously defined points. The conclusion can be any geometric property that can be expressed by the help of the defined geometric quantities involving only the constructed points. This method is also well-suited for computing geometric expressions that can be expressed using the same set of geometric quantities.

We use capital letters (or combination of capital letters and numbers) to denote points in the Euclidean plane.

We denote by $\bullet\mathrm{L}_{\{A,B\}}$ the length of the oriented segment from A to B and by $\bullet\mathrm{S}_{\{A,B,C\}}$ the signed area of the oriented triangle ABC. For an oriented quadrilateral $ABCD$, we define its area as $\bullet\mathrm{S}_{\{A,B,C,D\}} = \bullet\mathrm{S}_{\{A,B,C\}} + \bullet\mathrm{S}_{\{A,C,D\}}$.

In an oriented triangle ABC the Pythagorean difference $\bullet\mathrm{P}_{\{A,B,C\}}$ is defined as $\bullet\mathrm{P}_{\{A,B,C\}} = \bullet\mathrm{L}_{\{A,B\}}{}^2 + \bullet\mathrm{L}_{\{C,B\}}{}^2 - \bullet\mathrm{L}_{\{A,C\}}{}^2$.

We shall understand by geometric quantities the ratio of the length of two oriented segments on one line or on two parallel lines (denoted by $\bullet\mathrm{R}_{\{A,B,C,D\}}$), the signed area of an oriented triangle or a quadrilateral and the Pythagorean difference of an oriented triangle or a quadrilateral.

We use the elimination lemmas presented in [5].

4. The AreaCAD Method

Our goal is to prove constructive geometry statements possibly involving inequalities. As a first step, we reduce the original proof problem to an equivalent one that makes use only of the points that were introduced as free points. We compute the expressions that appear as additional hypothesis and conclusions eliminating the constructed points using the elimination steps of the area method [5]. Thus we obtain some expressions that depend on the free points and some rational constants denoted by r_i. These constants are introduced by the semibound points and appear when translating the original constructions into constructions accepted by the area prover.

Our original proof problem of finding nondegenerate conditions N such that

$$\begin{aligned}
&\underset{i=1,\ldots,p}{\underset{A_i}{\forall}}\ \underset{j=1,\ldots,m}{\underset{B_j}{\forall}}\quad point[A_1,\ldots,A_p]\wedge\\
&\qquad \wedge C_1[A_1,\ldots,A_p,B_1]\wedge\ldots\wedge C_m[A_1,\ldots,A_p,B_1,\ldots,B_m]\wedge\\
&\qquad \wedge H_1[A_1,\ldots,A_p,B_1,\ldots,B_m]\wedge\ldots\wedge H_n[A_1,\ldots,A_p,B_1,\ldots,B_m]\wedge\\
&\qquad \wedge N[A_1,\ldots,A_p,B_1,\ldots,B_m]\Rightarrow\\
&\qquad \Rightarrow G_1[A_1,\ldots,A_p,B_1,\ldots,B_m]\wedge\ldots\wedge G_k[A_1,\ldots,A_p,B_1,\ldots,B_m]
\end{aligned}$$

is transformed to the equivalent problem of finding nondegenerate conditions N' such that

$$\underset{\substack{A_i\\ i=1,\ldots,p}}{\forall}\ \underset{\substack{r_j\\ j=1,\ldots,q}}{\forall}\ H'_1[A_1,\ldots,A_p,r_1,\ldots,r_q]\wedge\ldots\wedge H'_n[A_1,\ldots,A_p,r_1,\ldots,r_q]\wedge$$
$$\wedge N'[A_1,\ldots,A_p,r_1,\ldots,r_q]\Rightarrow$$
$$\Rightarrow G'_1[A_1,\ldots,A_p,r_1,\ldots,r_q]\wedge\ldots\wedge G'_k[A_1,\ldots,A_p,r_1,\ldots,r_q]$$

As a second step we have to prove this statement using the CAD method. For the CAD algorithm we have to transform the obtained problem into polynomial form. To obtain as simple expressions as possible we choose for the origin of the coordinate system the point with the highest number of occurrences in the expressions. The next point is taken as being on the $x$axis. We may take this point as having coordinates $\{1,0\}$. This way the algebraic expressions become even simpler.

If the denominator of an obtained conclusion expression not being 0 does not result from the hypothesis this should be considered a non-degenerate condition and added to the hypothesis. At the end of the proof the user has to analyze whether the obtained condition is a non-degenerate condition or it introduces some essentially new hypothesis. If the simplified expressions contain square roots even powers of subexpressions should be extracted from the square roots adding the necessary conditions.

An implication is true if its conclusion is true or if the hypotheses are contradictory. In this second case we get no information on the logical value of the conclusion. Thus we check first the consistency of the hypothesis by an existential quantifier elimination, then the validity of the universally quantified expression is checked. For this purpose we use the built-in *Mathematica* functions.

5. Example

Let O be the circumcenter, I the incenter, G the centroid and H the orthocenter of a triangle. Then $OG \le OI \le OH$.

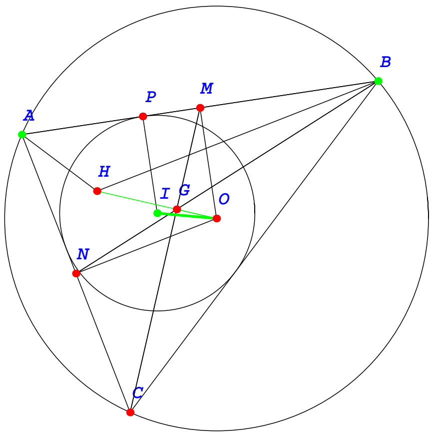
The input for the prover is a *Theorema* Proposition:

```
Proposition["Cad_113_27", any[I, A, B, C, P, H, M, N, O, G],
incircle[I, A, B, C, P] ∧ inter[H, tline[A, B, C], tline[B, A, C]] ∧
   median[C, M, A, B, C] ∧ median[B, N, A, B, C] ∧
   inter[O, tline[M, A, B], tline[N, A, C]] ∧ circle[O, A] ∧
   inter[G, line[C, M], line[B, N]]
   ⇒ inequation[seglengthsq[O, H] − seglengthsq[O, I] ≥ 0] ∧
      inequation[seglengthsq[O, I] − seglengthsq[O, G] ≥ 0]]
```

To display graphically the geometrical constraints among the involved points and lines we call function `Simplify`:

```
Simplify[Proposition["Cad_113_27"], by → GraphicSimplifier]
```

and obtain the output:



$\overline{HO}^2 \geq \overline{IO}^2$ for this configuration of the points
$\overline{IO}^2 \geq \overline{GO}^2$ for this configuration of the points

FIGURE 1. *Theorema* output

The geometry prover is invoked in the usual *Theorema* manner, specifying the AreaCAD prover, *Theorema* does the rest of the work:

– finds the convenient constructions recognized by the Area Prover;
– invoking the Area Prover computes the geometric expressions representing the constraints and conclusion;
– expresses the obtained new problem as a universally quantified boolean combination of polynomial equalities and inequalities, using the cartesian coordinates of the free points;
– invokes the *Mathematica* `ExistsRealQ` function to verify the consistency of the hypothsis, and then the `Resolve` function to find the trueness of the universally quantified formula;
– generates the notebook with all the explained details of the proof.

For the function call

```
Prove[Proposition["Cad_113_27"], by→GeometryProver, ProverOptions→
    {Method→"AreaCAD"}]
```

we obtain the following output from the prover:

=== *Begin of Theorema notebook* ===

We have to prove:

(Proposition(CAD_113_27))

$$\underset{I,A,B,C,P,H,M,N,O,G}{\forall}(\text{incircle}[I,A,B,C,P] \wedge \text{inter}[H,\text{tline}[A,B,C],\text{tline}[B,A,C]] \wedge \text{midpont}[M,A,B] \wedge \text{midpoint}[N,A,C] \wedge \text{inter}[O,\text{tline}[M,A,B],\text{tline}[N,A,C]] \wedge \text{circle}[O,A] \wedge \text{inter}[G,\text{line}[C,M],\text{line}[B,N]] \Rightarrow \text{inequation}[\text{seglengthsq}[O,H]-\text{seglengthsq}[O,I] \geq 0] \wedge \text{inequation}[\text{seglengthsq}[O,I]-\text{seglengthsq}[O,G] \geq 0]$$

with no assumptions.

As the proposition contains inequalities, we have to use the CAD method. We shall use the area method first to obtain a simpler input for CAD.

First we have to transform the problem in internal form for the AreaCAD. We have to prove, that constructions

$\{A,B,I\}$ free points

$\alpha_A B \perp AI$ and $\alpha_A \in AI$ with ndg. $A \neq I$

$\alpha 1_A B \parallel B\alpha_A$ and $\frac{\overline{B\alpha 1_A}}{\overline{B\alpha_A}} = 2$ with ndg. $B \neq \alpha_A$

$\alpha_B A \perp BI$ and $\alpha_B \in BI$ with ndg. $B \neq I$

$\alpha 1_B A \parallel A\alpha_B$ and $\frac{\overline{A\alpha 1_B}}{\overline{A\alpha_B}} = 2$ with ndg. $A \neq \alpha_B$

$C = A\alpha 1_A \cap B\alpha 1_B$ with ndg. $A \neq \alpha 1_A, B \neq \alpha 1_B, A\alpha 1_A \nparallel B\alpha 1_B$

$PI \perp AB$ and $P \in AB$ with ndg. $A \neq B$

$\alpha_C I \perp AC$ and $\alpha_C \in AC$ with ndg. $A \neq C$

$\gamma_H B \perp AC$ and $\gamma_H \in AC$ with ndg. $A \neq C$

$\beta_H A \perp BC$ and $\beta_H \in BC$ with ndg. $B \neq C$

$H = A\beta_H \cap B\gamma_H$ with ndg. $\beta_H \neq A, \gamma_H \neq B, \beta_H A \nparallel \gamma_H B$

$MA \parallel AB$ and $\frac{\overline{AM}}{\overline{AB}} = \frac{1}{2}$ with ndg. $A \neq B$

$NA \parallel AC$ and $\frac{\overline{AN}}{\overline{AC}} = \frac{1}{2}$ with ndg. $A \neq C$

$N\gamma_O \perp NA$ and $\frac{\overline{N\gamma_O}}{\overline{NA}} = r_{23}$ with ndg. $A \neq N$

$\beta_O \perp MA$ and $\frac{\overline{M\beta_O}}{\overline{MA}} = r_{22}$ with ndg. $A \neq M$

$O = M\beta_O \cap N\gamma_O$ with ndg. $\beta_O \neq M, \gamma_O \neq N, \beta_O M \nparallel \gamma_O N$

$G = CM \cap BN$ with ndg. $C \neq M, B \neq N, CM \nparallel BN$

with additional constraints

$\bullet R_{\{A,P,P,B\}} > 0$

$\bullet R_{\{A,\alpha_C,\alpha_C,C\}} > 0$

imply that $\overline{HO}^2 - \overline{IO}^2 \geq 0$ and $-\overline{GO}^2 + \overline{IO}^2 \geq 0$

Step 1

Now we try to obtain a simpler form of the constraints and of the conclusion using the area method.

Explanations

========= details about the area method =========

Expression 1

We have to compute $\bullet R_{\{A,P,P,B\}}$

========= details of the computation steps =========

We obtain:

$$\bullet R_{\{A,P,P,B\}} = E1 = \frac{\bullet L^2_{\{A,B\}} + \bullet L^2_{\{A,I\}} - \bullet L^2_{\{B,I\}}}{\bullet L^2_{\{A,B\}} - \bullet L^2_{\{A,I\}} + \bullet L^2_{\{B,I\}}}$$

Expression 2

We have to compute $\bullet R_{\{A,\alpha_C,\alpha_C,C\}}$

========= details of the computation steps =========

We obtain:

$$\bullet R_{\{A,\alpha_C,\alpha_C,C\}} = E2 = \frac{-\bullet L^4_{\{A,B\}} + \bullet L^4_{\{A,I\}} + 2\bullet L^2_{\{A,B\}}\bullet L^2_{\{B,I\}} - \bullet L^4_{\{B,I\}}}{\bullet L^4_{\{A,B\}} + \left(\bullet L^2_{\{A,I\}} - \bullet L^2_{\{B,I\}}\right)^2 - 2\bullet L^2_{\{A,B\}}\bullet L^2_{\{A,I\}} + \bullet L^2_{\{B,I\}}}$$

Expression 3

We have to compute $\overline{HO}^2 - \overline{IO}^2$

========= details of the computation steps =========

We obtain:

$\overline{HO}^2 - \overline{IO}^2 = \bullet P_{\{H,O,H\}} - \bullet P_{\{I,O,I\}} = E3 =$

$$\begin{aligned}
&-2(\bullet L^{12}_{\{A,B\}} - 4 \bullet L^{10}_{\{A,B\}} \bullet (L^2_{\{A,I\}} + \bullet L^2_{\{B,I\}}) + (\bullet L^2_{\{A,I\}} - \bullet L^2_{\{B,I\}})^4\\
&(\bullet L^4_{\{A,I\}} + \bullet L^4_{\{B,I\}}) + 7 \bullet L^8_{\{A,B\}}(\bullet L^4_{\{A,I\}} + \bullet L^4_{\{B,I\}} + \bullet L^2_{\{A,I\}} \bullet L^2_{\{B,I\}})-\\
&\bullet L^2_{\{A,B\}}(\bullet L^2_{\{A,I\}} - \bullet L^2_{\{B,I\}})^2\\
&(4 \bullet L^6_{\{A,I\}} + 4 \bullet L^6_{\{B,I\}} - \bullet L^2_{\{A,I\}} \bullet L^4_{\{B,I\}} - \bullet L^4_{\{A,I\}} \bullet L^2_{\{B,I\}})-\\
&\bullet L^6_{\{A,B\}}(8 \bullet L^6_{\{A,I\}} + 8 \bullet L^6_{\{B,I\}} + \bullet L^2_{\{A,I\}} \bullet L^4_{\{B,I\}} + \bullet L^4_{\{A,I\}} \bullet L^2_{\{B,I\}})+\\
&7 \bullet L^4_{\{A,B\}}(\bullet L^8_{\{A,I\}} + \bullet L^8_{\{B,I\}} - \bullet L^2_{\{A,I\}} \bullet L^6_{\{B,I\}} - \bullet L^6_{\{A,I\}} \bullet L^2_{\{B,I\}}))/\\
&(\bullet L^2_{\{A,B\}}(-\bullet L^2_{\{A,B\}} + \bullet L^2_{\{A,I\}} + \bullet L^2_{\{B,I\}})^2\\
&(\bullet L^4_{\{A,B\}} + (\bullet L^2_{\{A,I\}} - \bullet L^2_{\{B,I\}})^2 - 2 \bullet L^2_{\{A,B\}}(\bullet L^2_{\{A,I\}} + \bullet L^2_{\{B,I\}})))
\end{aligned}$$

Expression 4

We have to compute $-\overline{GO}^2 + \overline{IO}^2$

========= details of the computation steps =========

We obtain:

$-\overline{GO}^2 + \overline{IO}^2 = \bullet P_{\{I,O,I\}} - \bullet P_{\{O,G,O\}} = E4 =$

$$\begin{aligned}
&2(\bullet L^8_{\{A,B\}} - 2 \bullet L^6_{\{A,B\}}(\bullet L^2_{\{A,I\}} + \bullet L^2_{\{B,I\}}) + (\bullet L^2_{\{A,I\}} - \bullet L^2_{\{B,I\}})^2(\bullet L^4_{\{A,I\}} + \bullet L^4_{\{B,I\}})+\\
&\bullet L^4_{\{A,B\}}(2 \bullet L^4_{\{A,I\}} + 2 \bullet L^4_{\{B,I\}} - 7 \bullet L^2_{\{A,I\}} \bullet L^2_{\{B,I\}})+\\
&\bullet L^2_{\{A,B\}}(-2 \bullet L^6_{\{A,I\}} - 2 \bullet L^6_{\{B,I\}} + 11 \bullet L^2_{\{A,I\}} \bullet L^4_{\{B,I\}} + 11 \bullet L^4_{\{A,I\}} \bullet L^2_{\{B,I\}}))/\\
&(9 \bullet L^2_{\{A,B\}}(-\bullet L^2_{\{A,B\}} + \bullet L^2_{\{A,I\}} + \bullet L^2_{\{B,I\}})^2)
\end{aligned}$$

Step 2

Our initial problem becomes

$$\underset{A,B,I}{\forall}(E1 > 0 \wedge E2 > 0 \Rightarrow E3 \geq 0 \wedge E4 \geq 0)$$

Choosing a cartesian coordinate system with the x-axis $\{I, B\}$ and performing substitution: $\{x_I \to 0, y_I \to 0, x_B \to 1, y_B \to 0, x_A \to u_1, y_A \to u_2\}$ the problem becomes:

$$\begin{aligned}
&\underset{u_1,u_2}{\forall}\Big(\frac{-(-u_1+u_1^2+u_2^2)}{-1+u_1} > 0 \wedge \frac{u_1(u_1^2+u_2^2-u_1)}{u_2^2} > 0 \wedge u_1 \neq 0 \wedge u_2 \neq 0 \wedge -1 + u_1 \neq 0 \Rightarrow\\
&\frac{(u_1^8-u_1^7+6u_1^6+u_1^4(1-2u_2^4)+u_2^4(-1+u_2^2)^2-2u_1^2u_2^2-u_1(u_2^4+u_2^6)-u_1^5(4+u_2^2)+u_1^3u_2^2(3+2u_2^2))}{(u_1^2u_2^2(1+u_1^2+u_2^2-2u_1))} \geq 0 \wedge\\
&\frac{(-3u_1^5+4u_1^6+4u_2^4+u_1^2(4-6u_2^2+4u_2^4)+u_1^3(-3+2u_2^2)+u_1^4(-2+8u_2^2)+5u_1(u_2^2+u_2^4))}{(9u_1^2(1+u_1^2+u_2^2-2u_1))} \geq 0\Big)
\end{aligned}$$

Applying the CAD algorithm to this expression we obtain that the proposition is true.

=== *End of Theorema notebook* ===

6. Concluding Remarks

In this paper we presented a method for proving a class of plain Euclidean geometry theorems involving order relation. The algebraic methods for proving geometry theorems (Gröbner bases, characteristic sets) are very efficient, but they cannot deal with polynomial inequalities, that is geometry statemnts that involve order relation. On the other hand, Collins' CAD algorithm can deal with inequalities, but it is very slow for a system of polynomial equations and inequations with many variables (usually more than 15 for a nontrivial theorem). We propose a method that combines the point elimination used in the area method with Collins' CAD. We used our method to prove several quite difficult theorems, obtaining the result in reasonable time (at most some minutes).

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