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THE STABLE SETS OF A G-COMPLEX OF MULTI-ARY RELATIONS AND ITS APPLICATIONS

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Abstract. We define the notion of internally stable set and externally stable set for the G-complex of multi-ary relations. We define also a simple game with two players, where the solution is established by the kernel of G-complex of multi-ary relations.

The notion of complex of multi-ary relations was defined as a discrete structure, which consists of a special sequences of elements of a set X (see papers [1]-[3]). Generalizing some classical structure as graphs, complex of multi-ary relations is useful as a model to solve some problems in projection of integrating network, in informatics, economics etc. If the elements of the complex of multi-ary relations are sequences with repetitions we obtain a new structure called G-complex of multi-ary relations.

Let $X = (x_1, x_2, \dots, x_r)$ be a set of elements, $r \geq 2$, and $X = X^1$, X^2, $\dots$, X^n, $\dots$, $1 \leq n \leq r$, a sequence of cartesian products of the set X : $X^{m+1} = X^m \times X$, $m = 1, 2, \dots, n$. Any not empty subset $R^m \subset X^m$, $m \geq 1$, is said to be m-ary relation of elements from X. The set $R^1 \subset X^1$ define a subset of elements from X. The m-ary relation R^m consists of a family of sequence with m elements from X in a given order. Now, consider the finite subset of relations R^1, R^2, $\dots$, R^{n+1} of the infinite set mentioned above. Require that this subset satisfies the conditions:

I. $R^1 = X^1 = X$;

II. $R^{n+1} = \emptyset$;

III. Any sequence $[x_{j_1}, x_{j_2}, \dots, x_{j_l}]$, $1 \leq l \leq m \leq n+1$ of the sequence $[x_{i_1}, x_{i_2}, \dots, x_{i_m}]$, which represents sequence, is contained in R^l.

Definition 1. *A family of relations $R^1, R^2, \dots, R^{n+1}$ which satisfies the conditions I-III is said to be* ***finite-complex of multi-ary relations*** *and is denoted by $\mathcal{R}^{n+1} = (R^1, R^2, \dots, R^{n+1})$.*

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According to the conditions I-III it results that any set R^m of the complex of relations $\mathcal{R}^{n+1}$ is not empty.

By according to the study of G-complex of multi-ary relations in paper [4] is defined the basic notions of G-subcomplex, connectivity, path etc. A complex of multi-ary relations $\mathcal{R}^{n+1} = (R^1, R^2, \ldots, R^{n+1})$ can be represented as a family of abstract simplexes and is denoted by $K^n = (\mathcal{S}^0, \mathcal{S}^1, \ldots, \mathcal{S}^n)$ (see paper [4]).

Now let's examine the G-complex formed by the characteristic faces of the simplex $S^m \in K^n$, $m > 0$, which will be denoted $[S^m]$ and will be written $K_s^{m-1} = [S^m]$.

Definition 2. *Let K^n be a G-complex of multi-ary relations, an arbitrary simplex $S^m = [x_{i_0}, x_{i_1}, \ldots, x_{i_m}] \in K^m$, $m > 0$, and $[S^m]$ – the complex formed by S^m. The difference $S^m \backslash [S^m]$ will be called* ***vacuum*** *with the dimension m and will be notated by $\overset{\circ}{S^m} = S^m \backslash [S^m] = (x_{i_0}, x_{i_1}, \ldots, x_{i_m})$.*

Let's say F represents a set of simplexes from K^n. We will denote by T_F the family of all simplexes from K^n which does not contains the own face of simplexes of F, but are incident with F. We will denote by T_F the set of all vacuums (with different dimensions) of these elements (simplexes). Let's examine the difference $K^n \backslash (F \cup T_F)$. So, to simplify the notations, in the place of $K^n \backslash (F \cup T_F)$ we will use $K^n \backslash F$.

Denoted by $st\, S^m$ the set of all simplexes of dimension $m+1$, for which S^m is a common face, and all faced of these simplexes. The set $st\, S^m$ is said to be **star** of simplex S^m.

Let S^m be a simplex of G-complex of multi-ary relations K^n and K_s be a subcomplex of K^n that contains S^m, all proper faces of S^m, all simplexes of K^n incedented to S^n, and all his faces. The subcomplex $K_s \subset K^n$ is called **superstar** of S^m and is denoted by $St\, S^m$. The subcomplex $R(S^m) \subset K^n$ which consists of the set of all nonincidented simplexes to S^m of the superstar $St\, S^m$ is called **the representative of this superstar** and is denoted by $R(S^m)$.

Let's assume we have a connected G-complex of multi-ary relations K^n, an arbitrary simplex $S^m \in K^n$, a superstar $St(S^m)$ and a representative $R(S^m)$ of a superstar. We know that $R(S^m) \cap S^m = \emptyset$.

Now let's conseder a point to set mapping

$$\Gamma : K^n \to K^n,$$

with the property:

$$\Gamma(S^m) \subset R(S^m),$$

for all simplexes $S^m \in K^n$, $m = 0, 1, \ldots, n$.

Definition 3. *Let's say I is a family of simplexes of K^n that satisfies the condition: for any $S^m \in I$ it holds*

$$\Gamma(S^m) \cap I = \emptyset.$$

The family of simplexes I of the G-complex K^n is said to be ***internally independent set (internally stable set)*** *of simplexes of the G-complex K^n.*

Remark 1. *Not every G-complex of multi-ary relations possesses internally stable sets of simplexes relating to Γ, formed by the oriented complex*

$$S^m = \varepsilon(m)[x_{i_0}, x_{i_1}, \ldots, x_{i_m}] \in K^n.$$

Let K^n be a complex that admits independent sets, the J-ensemble of all shown sets of K^n.

Consequence 1. *If $I_1 \in J$ and $I_2 \subset I_1$, then we obtain the relation $I_\Gamma^2 \subset I_\Gamma$.*

Definition 4. *Let's consider K^n and the ensemble J. The value*

$$\alpha(K^n) = \max_{I \in J} \{card\, I\}$$

is to be said the ***internally independent number*** *of G-complex K^n (**the number of internally stability**).*

Definition 5. *Let K^n be a connected G-complex of multi-ary relation, and E a family of simplexes from K^n with the property: for any $S^m \in K^n \backslash E$ holds the relation $\Gamma(S^m) \cap E \neq \emptyset$. The family $E \subset K^n$ is called* ***externally stable set*** *(**externally stable set**) of simplexes of G-complex K^n.*

Let $\mathcal{E}$ be the ensemble of all externally stable sets of simplexes of complex K^n.

Consequence 2. *If $E_1 \in \mathcal{E}$, $E_2 \in \mathcal{E}$, where $E_1 \subset E_2$, then $E_2 \in \mathcal{E}$.*

Definition 6. *Let K^n be a complex of multi-ary relations, and E_Γ – a proper ensemble. The value*

$$\beta(K^n) = \min_{E \in \mathcal{E}} \{card\, E\}$$

is to be said the ***externally independent number*** *of G-complex K^n (**the number of externally stability**).*

Definition 7. *Let consider a connected G-complex of multi-ary relations K^n, application Γ, and $N \subset K^n$ – a family of simplexes with the properties:*

a) N is a set of internally stable simplexes;

b) N is a set of externally stable simplexes of K^n;

The family N is called ***kernel*** *of G-complex K^n.*

Theorem 1. *Let's consider a connected local complete G-complex of multi-ary relations K^n which admits the family of internally stable sets $J \neq \emptyset$. An element $I \in J$ is the kernel of complex K^n if and only if this one is maximal.*

Proof. Let $N \in J$ be the kernel of K^n and let's admit that this one is not maximal internally stable set of K^n. In this case there is at least one simplex $S^m \in K^n$ that satisfied the property $N \cup \{S^m\} = I \in J$ and $S^m \notin N$. By definition 7 (see the property a), if $I \in J$, we have $\Gamma(S^m) \cap I =$. On the other hand, in virtue of definition 7 (see the property b), using the relation $S^m \in K^n \backslash N$, it results the inequality $\Gamma(S^m) \cap N \neq \emptyset$, so $\Gamma(S^m) \cap I \neq \emptyset$.

Let's assume now that $I \in J$ is a maximal internally stable set of simplexes of the complex K^n, and let's demonstrate that I is a kernel of K^n too. We admit the opposite, there is a simplex $S^m \in K^n \backslash I$, such that $\Gamma(S^m) \cap I = \emptyset$. If this equality is satisfied, we obtain immediately that the family $I \cup \{S^m\}$ represents an internally stable set of simplexes of complex K^n, as $\Gamma(S^m) \cap S^m = \emptyset$. The contradiction with the assuming that $I \in J$ is maximal. The theorem 1 is proved.

Definition 8. *The next procedure is to be called* ***a simple game with the players** A **and** B **on the** G**-complex** K^n.*

1) A simplex $S^m \in K^n$ is determined arbitrary.

2) The player A picks up a simplex from the set $\Gamma(S^m)$ (if $\Gamma(S^m) = \emptyset$, the player A loses). Let S^{m_1} be the simplex picked up by A.

3) The player B picks up a simplex from $\Gamma(S^{m_1})$ (if $\Gamma(S^{m_1}) = \emptyset$, the player B loses). Let S^{m_2} be the simplex picked up by B.

4) and so on

Definition 9. *The sequence of simplexes $S^{m_0}, S^{m_1}, \ldots, S^{m_t}$ of G-complex K^n is to be called* ***trajectory*** *of K^n related to Γ, if the following conditions are satisfied:*

1) any pair of simplexes $S^{m_i}, S^{m_{i+1}}$ is particular, $i = 0, 1, \ldots, t-1$;

2) for $\forall S^{m_i} \in K^n$ it holds $S^{m_{i+1}} \subset \Gamma(S^{m_i})$, $i = 0, 1, \ldots, t-1$;

3) if $S^{m_{i+1}}$ and $S^{m_{i-j}}$ are intersected by a simplex $S^{m_{i+1,i-j}}$, where $j = 0, 1, \ldots, i-1$, then this simplex does not belong to the $S_{m_{i+1,i-j}}$.

The trajectory $S^{m_0}, S^{m_1}, \ldots, S^{m_t}$ will be denoted by $T(0,t)$.
The trajectory *$T(0,t)$ is said to be* ***maximal*** *related to Γ, if exists a natural number t, such that $\Gamma(S^{m_t}) = \emptyset$. The trajectory $T(0,t)$ is said to be* ***contour-trajectory*** *related to Γ, if exists t_0, such that $S^{m_0} = S^{m_{t_0}}$.*
Let's have a connected G-complex of multi-ary relations K^n, a series of nonnegative integers N_0 and a function $g : K^n \to N_0$, with the property: for any $S^m \in K^n$ it holds $g(S^m) = min\{N_0 \backslash g(\Gamma(S^m))\}$. The application g is to be called ***Grundy function*** *related to Γ.*

Not every G-complex K^n has a Grundy function.

It holds

Theorem 2. *Let K^n be a connected G-complex of multi-ary relations, and we consider it exists a Grundy function for K^n related to Γ. The set of all the* *simplexes S^m of K^n that satisfies the relation $g(S^m) = 0$ represents the kernel of complex K^n.*

Proof. Let M be the set of all the simplexes of K^n, which satisfies the relation $g(S^m) = 0$. We will show that:

1) M is an internal stable multitude of simplexes;
2) M is an external stable multitude of simplexes.

1) Let $S^{m_1} \in \Gamma(S^m)$ be a random simplex. From the definition of Γ and the application g, it results that $g(S^{m_1}) \neq 0$, that is $\Gamma(S^m) \cap M = \emptyset$.
2) Let $S^{m_1} \in K^n \backslash M$ be a random simplex. In this case $g(S^{m_1}) \neq 0$, so $\Gamma(S^m) \cap \cap M \neq \emptyset$ and M represents a kernel of the complex K^n.

The theorem 2 is proved.

Theorem 3. *If the G-complex K^n dispose of a contour-trajectory related to Γ, then the simple game on this G-complex leads to the fact that no one of the players A and B lose.*

The proof of theorem 3 is obvious because the players A and B choosing the simplexes on the contour-trajectory. Of cause, they always have the possibility to make the proper movement, if exists a simplex $S^{m_i} \in \Gamma(S^m)$, where S^m is an arbitrary simplex.

References

[1] Soltan P., *On the homologies of multi-ary relations and oriented hipergraphs*, Studii în metode de analiză numerică şi optimizare, vol. 2, nr 1(3), pp.60-81, Chişinău, 2000.

[2] Cataranciuc S., Soltan P., *Hipergrafuri şi omologiile lor*, Trends in the Development of the Information and Communication Technologies in Education and Management, International Conference, March 20-21, pp.294-300, 2003.

[3] Cataranciuc S., *About connectivity of general complexes of multi-ary relations*, The XIV-th Conference on Applied and Industrial mathematics (Satelite Conference of ICM 2006), pp.94-95, Chişinău, August 17-19, 2006.

[4] Cataranciuc S., *G-complexul de relaţii multi-are*, Analele ştiinţifice ale Universităţii de Stat din Moldova, Seria "Ştiinţe fizico-matematice", p.119-122, Chişinău, 2006.

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