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ON INDEPENDENT SETS OF VERTICES IN GRAPHS

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Abstract. Among the remarkable sets of vertices of a graph, the independent sets of vertices (acronym **IS**, other name is the internal stabile sets of vertices) are important, because using them we can solve many classification and counting problems: in chemistry, automobiles traffic, espionage, the chess game, the timetable problem. A new kind of independent sets of vertices fuzzy independent sets, related to fuzzy graphs are defined, and an algorithm for finding fuzzy independent sets (**FIS**) is given in this paper.

1. Introduction

There are several algorithms that determine the maximal **IS**-s of a graph. In [4] and [8], the authors have proposed an algorithm based on the calculus of associated boolean expressions of a graph. Another algorithm is due to Bednarek and Taulbee (which may be seen in [7]). A recursive version of this algorithm was given in [1]. Also, a simplified version of this algorithm that works with only one list was given there.

An algorithm based on the maximal complete submatrices was proposed by Y.Malgrange [5]. Using the Malgrange's relations a new and more clearly algorithm was given in [3].

These algorithms have an algebraic or a combinatorial character, and have $O(n^2)$ or $O(n^3)$ complexity.

2. Fuzzy independent sets of vertices in a fuzzy graph

Definition 2.1. A fuzzy graph is a graph $G = (V, \Gamma)$ with labeled edges. Every label is a number from $(0, 1]$ and shows the degree of the vertices association.

For a given fuzzy graph $G = (V, \Gamma)$ we can associate a fuzzy matrix (the degree of the association of every related vertex). See, for example, figure 1.

The fuzzy matrix is:

$$M = \begin{pmatrix} 0 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0 & 0.9 & 0 & 0 \\ 0 & 0.9 & 0 & 1 & 0.4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \end{pmatrix}$$

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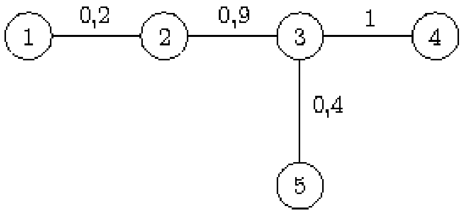

FIGURE 1

We can define the associated fuzzy matrix for a given graph:

$$M = (m_{ij}), \quad i = \overline{1,n},\ j = \overline{1,n},\ \text{with } m_{ij} = \begin{cases} \delta_{ij}, & [v_i, v_j] \in \Gamma \\ 0, & \text{otherwise} \end{cases}, \quad \delta_{ij} \in (0,1]$$

Definition 2.2. Let $G = (V, \Gamma)$ be a fuzzy graph. $S^\delta \subseteq V$ (δ is given too) is called a fuzzy independent set (**FIS**) of vertices iff $m_{ij} \leq \delta$ for each edge $[v_i, v_j]$, $v_i, v_j \in S^\delta$.

Remarks.

(1) We may define $G = (V, E)$ where E is the set of edges, $E \subseteq V \times V$; an edge is $[x, y]$, where $x, y \in V$ and we presume that E doesn't contain edges of the form $[x, x]$.
(2) Let S^δ be an **FIS**. We call S^δ maximal in relation to set inclusion, if S^δ is not included in any other **FIS**.
(3) Let S_G^δ be the family of maximal **FIS**-s of G. One can define:
   - $\alpha^\delta(G)$, the fuzzy internal stability number of the graph G as $\alpha^\delta(G) = \max |S^\delta|$, $S^\delta \in S_G^\delta$
   - $\gamma^\delta(G)$ is the fuzzy chromatic number of the graph G. If $S_1^\delta, \ldots, S_p^\delta$ are **FIS**-s with the properties: $S_i^\delta \cap S_j^\delta \neq \emptyset$, for $i \neq j$ and $\bigcup_{i=1}^{p} S_i^\delta = V$ than $S_1^\delta, \ldots, S_p^\delta$ form a chromatic decomposition of the graph G and $\Gamma(G) = \min(p)$. So $\Gamma^\delta(G)$ is the lowest number of disjoint **FIS**-s that covers G.

(4) $S_G^0 = S_G$.

For the graph depicted in figure 1 we have:
1) for $\delta = 1$ the FIS family is reduced to the entire set of vertices $\{1, 2, 3, 4, 5\}$
2) for $\delta = 0.5$ the FIS family is $\{\{1, 3, 5\}\{1, 2, 4, 5\}\}$
3) for $\delta = 0$ the set FIS family is $\{\{1, 3\}\{1, 4, 5\}\{2, 4, 5\}\}$ = IS family

3. A simplified algorithm to find the (FIS) of vertices using Malgrange's relations

Malgrange has defined the relations $\overline{\cup}$ and $\overline{\cap}$, as follows. Let M^* be the family of maximal submatrices of M (M is a boolean matrix). On the set M^* Malgrange define the $\overline{\cup}$ and $\overline{\cap}$ operations:

Let m_1 and m_2 be matrices of M^*, $m_1 = (A_1, B_1)$; $m_2 = (A_2, B_2)$.

A_1 contains row numbers, let i be one of them; B_1 contains column numbers, let j one of them; then $M(i, j) = 1$ (see the example below) then:

$$m_1 \overline{\cup} m_2 = (A_1 \cup A_2, B_1 \cap B_2) \quad \text{and} \quad m_1 \overline{\cap} m_2 = (A_1 \cap A_2, B_1 \cup B_2)$$

Example. Let G be the undirected graph (depicted in figure 1), $G = (V, E)$ where: $V = \{1, 2, 3, 4, 5\}$ and $E = \{[1, 2]; [2, 3]; [3, 4]; [3, 5]\}$.

The adjacent matrix is:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and the complementary matrix

$$\overline{M} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

A row-cover of $\overline{M}$ is $RC = \{(1, 1345); (2, 245); (3, 13); (4, 1245); (5, 1245)\}$.

A column-cover of $\overline{M}$ is $CC = \{(1345, 1); (245, 2); (13, 3); (1245, 4); (1245, 5)\}$.

Also $(1, 1345)\overline{\cup}(2, 245) = (12, 45)$ and $(1, 1345)\overline{\cap}(2, 245) = (\emptyset, 12345)$.

For a given fuzzy graph $G = (X, E)$ a row-cover from the associated fuzzy matrix, depending on δ, is defined by RC:

$$RC = (i, j_1, \dots, j_p) \in \text{with conditions } 0 < m_{i,j_1}, \dots, m_{i,j_p} \leq \delta \qquad (*)$$

The following algorithm computes the **FIS** of G.

S1. One finds C_0 a row-cover from the associated fuzzy matrix, depending of δ; $FFIS := \emptyset$; $k := 1$; (FFIS is the FIS family)

S2. a) One finds C_k:
 $\forall\, m_1, m_2 \in C_{k-1}$ (let $m_1 = (A_1, B_1)$ and $m_2 = (A_2, B_2)$)
 If $A_1 \cup A_2 \neq \emptyset$ and $A_1 \cup A_2 \subseteq B_1 \cap B_2$ then $m_1 \overline{\cup} m_2 \in C_k$.
 b) If $m \in C_k$ and $m = (I, I)$ then $FFIS := FFIS \cup I$ and $C_k := C_k \backslash m$;

S3. Repeat S2 for $k = 2, 3, , k_0$ until $C_{k_0} = \emptyset$.

Finally FFIS contains the FIS of G.

4. Example

An interesting problem is to predict the medical virus infection of certain locations (villages, towns). A graph $G = (V, E)$ is used to model this problem. V is the locations set and E represents the ways of virus propagation. Each arrow (v_i, v_j)

is labeled with δ_{ij} (the probability of virus propagation from v_i to v_j). At the beginning there are some infected locations. Let the vertex number 3 and vertex number 4 (see figure 1) be the first infected locations (denoted by $VF = \{3, 4\}$). We try to find out the future infected locations by with probability p (p is superior limit of the infection probability).

Problem solving steps are:

1. One finds out the FIS family for $\delta = 1 - p$. If $p = 0.7$ then $\delta = 0.3$ and the corresponding FIS family is

$$FFIS = \{\{1, 3, 4\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 4, 5\}\}.$$

From this family one chooses the FIS sets that include VF: $\{1, 3, 4\}$ and $\{2, 3, 4\}$.

2. From the previous sets one chooses the sets that include related vertices with every vertex from VF set. For our example the chosen set is $\{2, 3, 4\}$ (for the other set we can see that the vertex 1 is not related by vertex 2 and vertex 3 either. The problem is solved for the first period (an hour, a day, a week,...).

For the next period we'll chose $VF = \{2, 3, 4\}$ and we put the label 1 on the arrows between each pair from VF.

One can modify (if it is necessary) the labels of probability (between the vertices $V \setminus VF$) and on applying the previous steps (step 1 and step 2).

5. Conclusions

The above example tells us that the FIS family can solve many problems which are abstracted and modeled by the fuzzy graphs. So, we believe this kind of generalizations is useful to solve such problems.

A new kind of FIS can be obtained by modifying the RC definition given in section 3 $(*)$:

$$RC = (i, j_1 ... j_p) \in \text{with conditions } \delta \leq m_{i,j_1}, \ldots, m_{i,j_p} \leq 1.$$

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