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PREFIX-FREE LANGUAGES, SIMPLE GRAMMARS REPRESENTING A GROUP ELEMENT, LANGUAGES OF PARTIAL ORDER IN A GROUP

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Abstract. We show each word in the Kleene closure of a prefix-free language over an arbitrary alphabet has only one presentation as a concatenation of its words. It follows this language is the largest prefix-free one in the closure and both of them are simultaneously recursive or not recursive. We note if a complete simple grammar generates words only from a prefix-free language, the generated language exhausts it entirely. Such particular results only for word- and reduced word problem languages of a group can be found in [1, 4]. Using appropriate parts of the repeated in [4] construction from [1] we construct entire simple grammars whose terminal set is the monoid generating set of a group. The start symbol can be indexed by any element of the group and then the corresponding grammar will generate only representatives of this element. If all of them contain in its prefix-free part, their set exhausts this part according to the note above. Following [4] the reduced word problem language must be a such part of the word problem language for a group with finite irreducible word problem and the simple grammar there. The answer to the final question 5.3. [4] is absolutely analogous and simply follows the proof from [1]. We give necessary and sufficient conditions which a language must satisfy together with these [4] for a word problem language in order the first one to assign a partial order in the group of the second one.

1. Necessary Preliminary Concepts

We will use concepts from [1,4] in sections 2 and 3, and in addition from [2] in section 4. Our supplement is their concrete setting in an order, their connecting, and some formulations. A definition 1.1. for entire simple grammar is added below. We have corrected here the principle for a right inverse element as it is in [4] with the principle for a left inverse element because the requirement for a right one leads to another situation [3]. We note a generalization of the used in

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[4] syntactic congruence was used by us much earlier in [5], [6] for demonstration a pure subsemigroup of a group can not be covered by a regular language and for obtaining finite homomorphic images of some semigroups.

For Reading Section 2. For any set Σ let Σ^* denotes the set of all *finite words* in the elements of Σ, i.e. Σ^* is the set of all finite strings of these elements (Λ is the empty word). The number of the symbols in a such word is *its length.* The expression $s \equiv t$ means s and t are *identical* as strings of symbols. The word st is *a concatenation* of the word t after the word s. Any subset of Σ^* is *a language* in it. If L is a language, then *its Kleene closure* L^* in Σ^* is the language which consists exactly of all finite concatenations of the words from L. If $t \equiv us$, then u is said to be *a prefix* of t. It is *a proper prefix* if it is nonempty and ends before the end of t. Given a language L *the notation* $MIN(L)$ denotes the set of all words in L each one of which has no proper prefix in L. *A language* L is said to be *prefix-free* if $MIN(L) = L$. $MIN(L)$ is a prefix-free language. Analogically, about *suffix-free languages.*

A grammar is a four-tuple $\Gamma = (N, \Sigma, P, S)$ where N is the set of its *non-terminal symbols*, Σ is its set of *terminal symbols* all different from non-terminal ones, S is a non-terminal symbol of N called *start symbol*, and P is its *set of productions.* Each production has the form $\alpha \to \beta$ in which α and β are words from $(N \cup \Sigma)^*$ and α contains at least one non-terminal symbol. The word $\alpha_1\beta\alpha_2$ is *directly derived* from the word $\alpha_1\alpha\alpha_2$ by this production. Each sequence of direct derivations gives *a derivation* of its last word from the first one. *The language* $L(\Gamma)$ *generated by the grammar* $\Gamma = (N, \Sigma, P, S)$ is the set of all words over the terminal alphabet Σ which can be derived from the start symbol S. A grammar $\Gamma = (N, \Sigma, P, S)$ is *context-free* if each production has the form $A \to \beta$ where A is a non-terminal symbol. The generated by context-free grammars languages are *context-free languages.* Every such language can be generated by a grammar in Greibach normal form. *A context-free grammar is in Greibach normal form* if it contains no non-terminal symbols which do not participate in a derivation of some terminal word and if each production has one of the forms

$A \to aB_1B_2...B_n$,

$A \to a$, or

$S \to \Lambda$.

Here A is a non-terminal symbol, each B_i too and other than S, and a is a terminal symbol. A grammar in Greibach normal form is *simple* if for each non-terminal symbol and each terminal symbol no more than one production of the indicated form is allowed, i.e. if $A \to a\alpha$ and $A \to a\beta$ are productions, then $\alpha \equiv \beta$ and if $S \to \Lambda$ is a production, then it is only one. *A language is simple* if it can be generated by a simple grammar. In [1], lemma 2. *it is proven a simple language is prefix-free.* The author uses only *leftmost derivations* at each step of which the leftmost participation of a non-terminal symbol is separated: if $\alpha_1 A\alpha_2 \to \alpha_1\beta\alpha_2$ is a step in a leftmost derivation made by using the production $A \to \beta$, then α_1 is a terminal word, i.e. it contains only terminal symbols. The author shows *every word in a context-free language has a leftmost derivation* which can be obtained simply by changing the order in which productions are used in an arbitrary derivation.

Definition 1.1. *A simple grammar is entire if for each non-terminal and each terminal symbols it contains a production of the indicated above forms.* Then it contains only one production of the indicated forms for each pair of non-terminal and terminal symbols. Some of the above concepts will be used for reading the next section.

Other Concepts for Reading Section 3. In this section Σ will be *a double alphabet* $\Sigma = \{x_1, x_2, \ldots, x_n; \overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$. If a_i is x_i, then $\overline{a_i}$ is $\overline{x_i}$; if a_i is $\overline{x_i}$, then $\overline{a_i}$ is x_i $(i = 1, 2, \ldots, n)$. For each word $u \equiv a_{i_1} a_{i_2} \ldots\ a_{i_k}$ in Σ its *inverse word* $\overline{u}$ is $\overline{u} \equiv \overline{a_{i_k}} \ldots \overline{a_{i_2}}\ \overline{a_{i_1}}$ in which the symbols are inverse and set in the inverse order. Let G be a finitely generated group with a monoid generating finite set Σ (or with a group generating set $\Sigma^+ = \{x_1, x_2, \ldots, xn\}$ or $\Sigma^- = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$) and with a finite set of defining relators. Each element of the group G is presented by words from Σ^*. *The natural correspondence* φ *depicts* each generator a_i into the element $\varphi(a_i)$ of G which contains a_i. *This correspondence is extended inductively* for each word and so we receive *the natural homomorpphism* φ of Σ^* over G. *The word problem language of this group* is the full prototype $\varphi^{-1}(\imath)$ of the unite element $\imath$ in G. *The reduced word problem language* is its part of all words each one of which does not have a proper prefix in it. If we denote the word problem language with $\mathcal{E}$ ($\mathcal{E} = \varphi^{-1}(\imath)$) and the reduced word problem language with $\mathcal{R}$, then $\mathcal{R} = MIN(\mathcal{E})$ according to the above notation of the function MIN. *The irreducible word problem language* is the set of all words from the (reduced) word problem language which have no proper subwords from the word problem language. It is not especially necessary to note these languages are shortly correspondingly named word problem, reduced word problem, and irreducible word problem only in [1, 4]. We prefer the terms with an added word "language" because they are language interpretations related to the word problem in the theory of groups.

Additional Concepts for Reading Section 4. *A group is partially ordered* if there is a partial order $\geq$ in it which is concerted with the group operation. That means if $a \geq b$ in the group, then $ax \geq bx$ and $xa \geq xb$ in it. The multiplication preserves the strong inequequality. The set of all strongly positive elements (i.e. of the elements which are strongly bigger than the unite element) is called *a strongly positive cone of the partially ordered group* and it is *a pure subsemigroup* of this group. That means it is an invariant subsemigroup which does not contain the unite element. Conversely, each pure subsemigroup assigns a partial order in the corresponding group.

2. Kleene Closures of Prefix-Free Languages in an Arbitrary Alphabet and Entire Simple Grammars

In this section the alphabet Σ is arbitrary. The general properties of the prefix-free languages below are induced by the properties of word problem and reduced word problem languages.

Proposition 2.1. *Let R be a prefix-free language in Σ^* and R^* is its Kleene closure. Then*

(1) Each word from R^* has only one presentation as a concatenation of words from R;

(2) $MIN(R^*) = R$;

This property can be expressed in an equivalent form:

(2') R is the largest prefix-free language in R^* (i.e. there exists no a prefix-free extension of R in R^*). R^* by itself is not prefix-free of course.

Proof. (1) If the word w from R^* is empty ($w \equiv \Lambda$), there is nothing to prove. Let w is a nonempty word from R^* and it has two presentations as concatenations of nonempty words from R:

$w \equiv v_1 v_2 ... v_k$, where $v_1, v_2, ..., v_k \in R$, and

$w \equiv w_1 w_2 ... w_l$, where $w_1, w_2, ..., w_l \in R$.

We have to prove $k = l$ and $v_1 \equiv w_1$, $v_2 \equiv w_2$, ..., $v_k \equiv w_k$. The proof is inductive with respect to the sum $k + l$ of the numbers of the factors in these presentations. Its minimal value is 2. Then $w \equiv v_1 \equiv w_1$ and the statement is obvious. Let it be true for all natural numbers whose sum is less than $k+l$. The first factors v_1 and w_1 from the prefix-free language R in the indicated presentations of w coincide because each one of them can't be a proper prefix of the other one. Therefore we have the presentations $w' \equiv v_2 ... v_k \equiv w_2 ... w_l$ of the remaining part w' of the word w after v_1 ($v_1 \equiv w_1$). The sum of the numbers of the factors in these presentations is $k + l - 2$. According to the inductive conjecture we receive $k - 1 = l - 1$ and $v_2 \equiv w_2$, ..., $v_k \equiv w_k$ for the all next factors.

(2) $MIN(R^*)$ is the subset of all words in R^* which one of which has no proper prefix in it (R^*). Each word in R has no proper prefix in itself. It follows from this each such word can not have a proper prefix in R^* because every word in R^* is a concatenation of words from R and, then, it would follow its first factor from R would be a proper prefix of a word in R. This is impossible because R is a prefix-free language. Therefore $R \sqsubseteq MIN(R^*)$. Conversely, each word in $MIN(R^*)$ can not have more than one factor in its presentation as a concatenation of R-words because, in the opposite case, its first factor would be a proper prefix of its in R^*. Therefore $MIN(R^*) \sqsubseteq R$ and $MIN(R^*) = R$.

Our idea for the Kleene closure of an arbitrary prefix-free language comes from [1,4] where the authors prove statements for word problem and reduced word problem languages in a double alphabet only:

Proposition 3.1. [4] *The word problem of a group with respect to a monoid generating set is the Kleene closure of its reduced word problem with respect to that generating set.*

Proposition 3.2. [4] *If W is the word problem of a group with respect to a monoid generating set and R is the reduced word problem with respect to this set, then $R = MIN(W) \cap X^+$.*

Lemma 4. [1] *If π is a finitely generated group presentation and L is prefix-free language such that $WP_0(\pi) \sqsubseteq L \subset WP(\pi)$, then $L = WP_0(\pi)$.* In the notations of the author there $WP(\pi)$ is the word problem (language) of π and $WP_0(\pi)$ is its reduced word problem (language).

The property (1) is nowhere else indicated and obviously it is very important at all due to the universal only one presentation. It has an application for the proof of the simultaneous recursiveness below. Property (2) (indicated in [4], proposition 3.2, for reduced word problem and word problem languages only) is important due to an unification of different requirements to the prefixes in the definitions and, in addition, then it is not necessary to show its equivalent form (2') as a property which is separated from the presentation, as this is done above in [1], lemma 4.

Corollary 2.2. *If R is a prefix-free language in Σ^* and R^* is its Kleene closure, then both of them are simultaneously recursive or not recursive.*

The proof is practically the same as in [4], theorem 3.5., where the formulation and the proof are again for reduced word problem and word problem languages in a double alphabet only, but using the previous proposition of ours here and without passing through the recursive enumerating of R.

Let R be recursive in Σ^*, i.e. there exists an effective procedure $\mathcal{A}$ for recognizing whether a word is from R or not. We will show then there exists an effective procedure $\mathcal{B}$ for recognizing belonging of any word to R^* without passing through the recursive enumerating of R as it is in [4]. Let w is an arbitrary word from Σ^*. We apply the algorithm $\mathcal{A}$ to its beginning. If $\mathcal{A}$ stops at some prefix of w showing this prefix is from R, we denote it with v_1, i.e. $w \equiv v_1 w_1$, where $v_1 \in R$. In the opposite case if $\mathcal{A}$ passes the entire word w with an answer it does not belong to R, then the algorithm $\mathcal{B}$ answers w does not belong to R^* because every word from R^* is a concatenation of words from R. In the first case if w_1 is empty ($w_1 \equiv \Lambda$) the algorithm $\mathcal{B}$ answers $w \equiv v_1$ belongs to R and therefore it belongs to R^*. If w_1 is not empty, we apply the algorithm $\mathcal{A}$ to w_1 for which we will have two analogous cases. In the first one $w \equiv v_1 v_2 w_2$ where v_1 and v_2 belong to R ($v_1, v_2 \in R$). In the opposite one if $\mathcal{A}$ passes the entire word w_1 with an answer it does not belong to R, then the algorithm $\mathcal{B}$ answers w does not belong to R^* because every word from R^* is a concatenation of words from R.

This inductive process is finite because every next applying the algorithm $\mathcal{A}$ is to a shorter word. A very important note is this applying is in only one way which is determined by the property (1) from proposition 1. above: each word from R^* has only one presentation as a concatenation of words from R.

We will receive in this way finally a single presentation $w \equiv v_1 v_2 \ldots\ v_k w_k$ (where v_1, v_2, $\ldots$, v_k belong to R, i.e. $v_1, v_2, \ldots, v_k \in R$) for which a next applying the algorithm $\mathcal{A}$ can not separate a prefix from R in w_k. Therefore w_k is empty ($w_k \equiv \Lambda$) in the first case or w_k does not belong to R in the opposite second case when $\mathcal{A}$ passes it entirely with an answer it is not from R. In the first case the algorithm $\mathcal{B}$ gives an answer the word $w \equiv v_1 v_2 \dots v_k$ belongs to R^*. In the second one its answer is the word $w \equiv v_1 v_2 \dots v_k w_k$ does not belong to R^* because every word from R^* is a concatenation of words from R and there isn't a way to receive a such concatenation for w due to its only one presentation as a possible such concatenation according to the property (1). Therefore $\mathcal{B}$ is an effective procedure for recognizing belonging of any word from Σ^* to R^*, i.e. if R is recursive, then R^* is recursive too.

Conversely, let R^* is recursive. The proof R is recursive is the same as in [4] but based on the property (2) from proposition 1.: $MIN(R^*) = R$. We have an algorithm for recognizing membership of R^*. We test a given word and all its proper prefixes for this membership. According to the definition of the function MIN and the indicated property this word is from R if and only if when it belongs to R^* but no proper prefix of its belong to R^*. So R is recursive.

Lemma 2.3. *Let $\Gamma = (N, \Sigma, P, S)$ be a complete simple grammar which generates only words from the prefix-free language R in Σ^*. Then the generated by it language $L(\Gamma)$ covers the entire R, i.e. $L(\Gamma) = R$.*

Proof. Let w is an arbitrary word from R. We have to prove it can be generated by Γ. Due to the completeness of Γ (Definition 1.1) there are enough productions in P to continue a leftmost derivation of w. We will show no one derivation can end before or after the end of w. If a derivation ends before the last letter of w, then the derived proper prefix of its belongs to R which is in a contradiction with the given fact R is a prefix-free language. If a derivation ends after the last letter of w, then w from R would be a proper prefix of the derived word again from R which is again impossible. Therefore each derivation in Γ, which starts from the first letter of w, ends immediately after its last letter and therefore $w \in L(\Gamma)$.

A particular case of this statement is practically proved in [1], lemmas 5-8, but in a very long way and again for reduced word problem language of a group only. Probably this way has been a reason the proof to be repeated in the second part of the proof of lemma 5.1. [4] , but briefly inductively with respect to the length of the word and again for this particular language only. We will note we don't need here the property the generated by a simple grammar (simple) language is prefix-free in an arbitrary alphabet which is proved in [1], lemma 2. The above lemma 2.3. can be formulated in the following more expressive form:

Lemma 2.3'. *No one complete simple grammar can generate an absolute part of a prefix-free language.*

3. Entire Simple Grammars Generating Only Representatives of Any Fixed Element of a Group

In this section Σ is a double alphabet $\Sigma = \{x_1, x_2, \dots, x_n; \overline{x_1}, \overline{x_2}, \dots, \overline{x_n}\}$. Let G be a finitely generated group with a monoid generating finite set Σ and with a finite set of defining relators. Let a be either one of the generators x_i or $\overline{x_i}$ ($a \equiv x_i$ or $a \equiv \overline{x_i}$, $i = 1, 2, \dots, n$), but $\mathfrak{a}$ is the element of G whose a representative is

a (the symbol a is from Σ, the element $\mathfrak{a}$ is from G). We will construct many complete simple grammars with one and the same set Σ of terminal symbols, with different sets of non-terminal symbols, different initial symbols, and different sets of productions which will be assigned in one and the same way. The initial idea comes from [1], lemma 15, it was used in the same way in [4], lemma 5.1. Our modifications will be shown after the next proposition.

The basic part of the non-terminal symbols will be the set $\{..., n_{\mathfrak{a}}, n_{\mathfrak{a}^{-1}}, ...\}$ of $2n$ symbols which correspond to the $2n$ elements $\mathfrak{a}$ and $\mathfrak{a}^{-1}$ of the group with representatives the generators a and $\bar{a}$. We add to them a set $\{n_{\mathfrak{g}'}\}$ of symbols $n_{\mathfrak{g}'}$ which correspond to at one's own choosing chosen finite number other elements $\mathfrak{g}'$ of this group. The set $N_{\{\mathfrak{g}\}}$ of the non-terminal symbols will be their union, i.e. $N_{\{\mathfrak{g}\}} = \{..., n_{\mathfrak{a}}, n_{\mathfrak{a}^{-1}}, ...\} \cup \{n_{\mathfrak{g}'}\} = \{n_{\mathfrak{g}}\}$.

We construct an obviously complete system P of simple productions in the following way: *for every* $n_{\mathfrak{g}} \in N_{\{\mathfrak{g}\}}$ *and every* $a \in \Sigma$

$n_{\mathfrak{g}} \to a$ *is a production if* $\mathfrak{g} = \mathfrak{a}$;
$n_{\mathfrak{g}} \to a\ n_{\mathfrak{a}^{-1}\mathfrak{g}}$ *is a production if* $\mathfrak{g} \neq \mathfrak{a}$ *and* $n_{\mathfrak{a}^{-1}\mathfrak{g}} \in N_{\{\mathfrak{g}\}}$;
If $n_{\mathfrak{1}}$ ($\mathfrak{g} = \mathfrak{1}$) is a terminal symbol, all productions of the form
$n_{\mathfrak{1}} \to a n_{\mathfrak{a}^{-1}}$
are among them because $\mathfrak{a} \neq \mathfrak{1}$
and $n_{\mathfrak{a}^{-1}.\mathfrak{1}} = n_{\mathfrak{a}^{-1}} \in N_{\{\mathfrak{g}\}}$.
$n_{\mathfrak{g}} \to a\ n_{\mathfrak{a}^{-1}} n_{\mathfrak{g}}$ *is a production if* $\mathfrak{g} \neq \mathfrak{a}$ *and* $n_{\mathfrak{a}^{-1}\mathfrak{g}} \notin N_{\{\mathfrak{g}\}}$.

The initial start symbol S of a such complete simple grammar $\Gamma = (N_{\{\mathfrak{g}\}}, \Sigma, P, S)$ can be any non-terminal symbol $n_{\mathfrak{g}_0}$, i.e. $S = n_{\mathfrak{g}_0}$ ($\mathfrak{g}_0$ is one of all participating elements $\mathfrak{g}$ of the group G).

Proposition 3.1. *All words derived from each non-terminal symbol* $n_{\mathfrak{g}}$ *of the just constructed complete simple grammar* $\Gamma = (N_{\{\mathfrak{g}\}}, \Sigma, P, S)$ *are representatives of the element* $\mathfrak{g}$ *in the group* G *with which* $n_{\mathfrak{g}}$ *is indexed.*

The proof is by induction with respect to the length of the derivation and repeats the first part of the proof of lemma 5.1 [4] and this of lemma 15 [1] but for more general grammars. Any derivation of length one in Γ is of the type $n_{\mathfrak{g}} \to a$ and by definition if $\mathfrak{g} = \mathfrak{a}$ only (a is a representative of $\mathfrak{g}$, i.e. $a \in \mathfrak{g}$). Let us assume the statement is true for all derivations of lengths no more than m, $m \geq 1$, starting from any non-terminal symbol. Any derivation from $n_{\mathfrak{g}}$ of length $m+1$ can start with one of the productions:

$n_{\mathfrak{g}} \to a\ n_{\mathfrak{a}^{-1}\mathfrak{g}}$ (*if* $\mathfrak{g} \neq \mathfrak{a}$ *and* $n_{\mathfrak{a}^{-1}\mathfrak{g}} \in N_{\{\mathfrak{g}\}}$) or
$n_{\mathfrak{g}} \to a n_{\mathfrak{a}^{-1}} n_{\mathfrak{g}}$ (*if* $\mathfrak{g} \neq \mathfrak{a}$ *and* $n_{\mathfrak{a}^{-1}\mathfrak{g}} \notin N_{\{\mathfrak{g}\}}$).

The lengths of the derivations which continue after those productions are no more than m and we can apply the inductive conjecture to them. So, in the first case the non-terminal $n_{\mathfrak{a}^{-1}\mathfrak{g}}$ on the right hand side derives a word u which belongs to $\mathfrak{a}^{-1}\mathfrak{g}$. The word au which will be derived from $n_{\mathfrak{g}}$ will belong to the element $\mathfrak{a}\mathfrak{a}^{-1}\mathfrak{g} = \mathfrak{g}$ of G, i.e. $au \in \mathfrak{g}$.

In the second case $n_{\mathfrak{a}^{-1}}$ on the right hand side derives a word s which is from $\mathfrak{a}^{-1}$ in G. The other non-terminal symbol $n_{\mathfrak{g}}$ there derives a word t which is from $\mathfrak{g}$. (The sum of the lengths of both last derivations is m.) The word ast which will be derived from $n_{\mathfrak{g}}$ will belong to $\mathfrak{a}\mathfrak{a}^{-1}\mathfrak{g}$ in G, i.e. $ast \in \mathfrak{a}\mathfrak{a}^{-1}\mathfrak{g} = \mathfrak{g}$.

This completes the inductive proof of the proposition.

$MIN(\mathfrak{g})$ is the set of all prefix free words from the element $\mathfrak{g}$ of the group G, i.e. the set of all words from $\mathfrak{g}$ which one of which has no proper prefix in it. The grammar from proposition 1. above is simple and complete. Therefore according to lemma 2.3. from section 2. this proposition has the following

Corollary 3.2. *For each element $\mathfrak{g}_0$ of the group G such that $n_{\mathfrak{g}_0}$ is a start symbol of the complete simple grammar $\Gamma = (N_{\{\mathfrak{g}\}}, \Sigma, P, S = n_{\mathfrak{g}_0})$ if the generated by this grammar representatives of $\mathfrak{g}_0$ contain in $MIN(\mathfrak{g}_0)$, then $L(\Gamma)$ coincides with the set $MIN(\mathfrak{g}_0)$ of all words from $\mathfrak{g}_0$ which have no proper prefix in it, i.e. $L(\Gamma) = MIN(\mathfrak{g}_0)$; briefly: if $L(\Gamma) \sqsubseteq MIN(\mathfrak{g}_0)$, then $L(\Gamma) = MIN(\mathfrak{g}_0)$.*

In particular, if $\mathfrak{g}_0 = \imath$ (then the start symbol is $S = n_\imath$), all words derivable by $\Gamma = (N_{\{\mathfrak{g}\}}, \Sigma, P, S = n_\imath)$ are from the word problem language of the group G. The simple grammars from lemma 15 [1] and lemma 5.1 [4] for theorem 5.2 [4] are constructed over a finite irreducible word problem language. They satisfy the conditions from corollary 3.2. here and we would have the proved there property from the indicated lemmas as a

Corollary 3.3. (**Theorem 5.2.** [4]) *If a finitely generated group has finite irreducible word problem language, then it has simple reduced word problem language.*

NOTE: We will show some of the modifications which we promised in the beginning of this section. First one is separating the elements $\mathfrak{a}$ and a^{-1} of the group whose representatives are the inverse generators a and $\overline{a}$. It is not necessary to prove especially a and $\overline{a}$ have inverse words, the corresponding products with which are from the irreducible word problem language of the group. That is correct simply because each one of them is obviously inverse to the other one and the words $a\overline{a}$ and $\overline{a}a$ are obviously irreducible. We can add arbitrary elements of the group to them and the same proof for generating only the indicated representatives goes. So, these modifications are significant because they lead to the just pointed freedom.

Statement 3.4. *Each group with a simple reduced word problem language with respect to some monoid generating set has a finite irreducible word problem language with respect to this generating set.*

This statement is an answer to question 5.3. with which the paper [4] ends. For its proof the notations of the type u^{-1} in lemmas 9. and 10. from [1] must simply be substituted by notations of the type $\overline{u}$ where the word $\overline{u}$ is the inverse word of u in a double alphabet.

4. Languages of Partial Order in a Group

Let G be a (monoid) finitely generated group with a generating set the double alphabet Σ as in the previous section. Let the partial order in G be presented by its positive cone $P = P^+ \cup \{\imath\}$. Here P^+ is the pure subsemigroup of the strongly positive elements, i.e. it is an invariant subsemigroup of the group which does not contain the unit element $\imath$. Let $\mathcal{P}$ and $\mathcal{P}^+$ be the full prototypes of P and P^+ correspondingly at the natural homomorphism φ of Σ^* over G, i. e. $\varphi^{-1}(P) = \mathcal{P}$ and $\varphi^{-1}(P^+) = \mathcal{P}^+$. Let $\mathcal{E} = \mathcal{P} \cap \overline{\mathcal{P}}$. Then $\varphi^{-1}(\imath) = \mathcal{E}$ (i.e. $\mathcal{E}$ is the word problem language of the group G). We can name $\mathcal{P}$ *a language of the positive cone in* G or *a positive cone language*.

The word problem language $\mathcal{E}$ in Σ^* satisfies two conditions from

Proposition 3.3. [4] *Let* $\mathcal{E}$ *be a subset of* Σ^**; then* $\mathcal{E}$ *is the word problem of a group if and only if it satisfies the following conditions:*

(1) if $\alpha \in \Sigma^*$*, then there exists* $\beta \in \Sigma^*$ *such that* $\beta\alpha \in \mathcal{E}$ ($\alpha\beta \in \mathcal{E}$ stays here incorrectly in [4])*;*

(2) if $\alpha \in \mathcal{E}$ *and* $u\alpha v \in \mathcal{E}$*, then* $uv \in \mathcal{E}$*.*

We denote here the word problem language by $\mathcal{E}$ instead by W as it is in [4]. The reason for the marked correction is indicated in the section for the preliminary concepts.

Theorem 4.1. *Let* G *be a finitely generated group with a monoid generating set* Σ *and* $\mathcal{E}$ *be its word problem language which satisfies the indicated above conditions (1) and (2). If* P^+ *is the cone of the strongly positive elements of some partial order in* G*, then its full prototype* $\mathcal{P}^+ = \varphi^{-1}(P^+)$ *at the natural homomorphism* φ *satisfies the following conditions:*

(3) if $uv \in \mathcal{P} = \mathcal{P}^+ \cup \mathcal{E}$ *and* $\alpha \in \mathcal{P}$*, then* $u\alpha v \in \mathcal{P}$*;*

(4) if $\alpha \in \mathcal{P}^+$*, then every* β *from (1), for which* $\beta\alpha \in \mathcal{E}$*, does not belong to* $\mathcal{P}^+$*.*

Conversely, if some language $\mathcal{P}^+$ *in* Σ^* *for which* $\mathcal{P}^+ \cap \mathcal{E} = \varnothing$ *satisfies the conditions (3) and (4), then it assigns a partial order in the group* G *with a word problem language* $\mathcal{E}$*.*

Proof. Let G be a partially ordered group with generators Σ and with a cone P of the positive elements. Then the corresponding language of the positive cone in Σ^* is $\mathcal{P} = \varphi^{-1}(P)$. The language $\mathcal{E} = \mathcal{P} \cap \overline{\mathcal{P}}$ is exactly the word problem language of the group G and it satisfies the conditions (1) and (2). Let $uv \in \mathcal{P}$ and $\alpha \in \mathcal{P}$. Then $\varphi(uv) = \varphi(u)\,\varphi(v) = \mathfrak{uv} \in P$, i.e. $\mathfrak{uv} \geq \imath$ and $\varphi(\alpha) = \mathfrak{a} \in P$. From $\mathfrak{a} \geq \imath$ we receive $\mathfrak{uav} \geq \mathfrak{uv}$ and therefore $\mathfrak{uav} \geq \imath$. The last one means $uav \in \mathcal{P}$ because $\varphi(uav) = \mathfrak{uav}$ with which the condition (3) is proved. The product of two strongly positive elements is again strongly positive from which it follows no one of the factors (in (4), where we suppose the opposite) can be inverse to the other one. This proves the last condition.

Conversely, suppose the language $\mathcal{P}^+(\mathcal{P}^+ \cap \mathcal{E} = \varnothing)$ in Σ^* together with $\mathcal{E}$ satisfies the conditions (3) and (4). We construct a group G whose word problem language is $\mathcal{E}$ according to the indicated proposition from [4]. We will show the image $P^+ = \varphi(\mathcal{P} \setminus \mathcal{E})$ of $\mathcal{P}^+$ is a pure subsemigroup in G, i.e. P^+ assigns a partial order in it. We note before that the used in this proposition from [4] syntactic congruence

$\alpha_1 \sim \alpha_2 \Leftrightarrow (ua_1v \in \mathcal{E} \Longleftrightarrow u\alpha_2 v \in \mathcal{E}$ for arbitrary $u, v \in \Sigma^*)$ could be found in the cited in [5], [6] paper by Rabin M.D. and D. Scott and monograph by S. Ginsburg. It was shown in the section with preliminaries why a generalization of its was introduced and used by us. *in them.*

Two corollaries from the property (3) are valid: (3a) if $\alpha, \beta \in \mathcal{P}$, then $\alpha\beta \in \mathcal{P}$ and (3b) for every $u \in \Sigma^*$ and every $\alpha \in \mathcal{P}$ the conjugate word $\overline{u}\alpha u \in \mathcal{P}$. Really, (3a): For the empty word Λ and β the word $\Lambda\beta \in \mathcal{P}$. Then $\Lambda\alpha\beta \equiv \alpha\beta \in \mathcal{P}$; (3b) $\overline{u}u \in \mathcal{E}$ and $\alpha \in \mathcal{P}$. Then $\overline{u}u \in \mathcal{P}$ and $\overline{u}\alpha u \in \mathcal{P}$ after (3).

Our goal is to prove $P^+ = \varphi(\mathcal{P}^+)$ is a pure subsemigroup in the already constructed group G. Let $\mathfrak{a}, \mathfrak{b} \in P^+$. Then there exist prototypes $\alpha \in \mathcal{P}^+$, $\beta \in \mathcal{P}^+$ of theirs and $\alpha\beta \in \mathcal{P}$ due to (3a). It $(\alpha\beta)$ can't be from $\mathcal{E}$ due to the condition (4).

We will show $P^+ = \varphi(\mathcal{P}^+)$ is invariant. Let $\mathfrak{a}$ is an arbitrary element of P^+ and $\mathfrak{u}$ is an arbitrary element of the entire group G. Let $\alpha \in \mathcal{P}^+$ is a prototype of $\mathfrak{a}$ and $u \in \Sigma^*$ is a prototype of $\mathfrak{u}$. Then $\overline{u}u \in \mathcal{E}$ and from $\overline{u}u \in \mathcal{P}$ and $\alpha \in \mathcal{P}$ we receive $\overline{u}\alpha u \in \mathcal{P}$. Therefore $\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u} = \varphi(\overline{u}\alpha u) \in P$. It remains to prove $\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u} \neq \imath$. Really, if $\mathfrak{u}^{-1}\mathfrak{a}\mathfrak{u} = \imath$, then $\mathfrak{a} = \imath$ which contravenes $\mathfrak{a} \in P^+$.

The theorem is proved. We would like to pay attention to the condition (3). Obviously it is reversed to the condition (2) cited from [4] where it is proved (as a consequence of its) a word problem language satisfies (3) too. It would be interesting to prove or to disprove a

Conjecture 4.2. *The condition (2) is a consequence from the condition (3) in* $\mathcal{E}$.

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