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ON TUBULAR SURFACES IN COMPUTER GRAPHICS

PAUL A. BLAGA

Abstract. We propose new approaches to the investigation of tubular and canal surfaces, regarded as swept surfaces, we give a parametric representation of the inverse of a canal surface and we suggest several applications of tubular surfaces in scientific vizualization.

1. Introduction

Tubular surfaces are among the surfaces which are easier to describe both analytically and “operationally”. They are still under active investigation, both for finding best parameterizations (see, for instance, [9, 11]) or for application in different fields (for instance in medicine, see [6]).

We remind that, if C is a space curve, a *tubular surface* associated to this curve is a surface swept by a family of spheres of constant radius (which will be the radius of the tube), having the center on the given curve. Alternatively, as we shall see in the next section, for them we can construct quite easily a parameterization using the Frenet frame associate to the curve. The tubular surfaces are used quite often in computer graphics, but we think they deserve more attention for several reasons. For instance, there is the problem of representing the curves themselves. Usually, the space curves are represented by using *solids* rather then tubes. There are, today, several very good computer algebra system (such as Maple, or Mathematica) which allow the vizualisation of curves and surfaces, in different kind of representations. However, the graphical output they produce is not always of the best quality and it is, definitely, useful to be able to export the graphic in a format which is recognized by a professional software, such as RealStudio, or AutoCad, which, in turn, have superior rendering possibilities. Unfortunately, the DXF format, for instance, is based on meshes, therefore, for instance, if one produces a graphic containing both curves and surfaces, the curves will be “lost in translation” when the graphic is exported as a DXF file. The situation would be different, of course, if one would be able to choose to represent the curves as tubes, rather then solids. Another problem is related to the flexibility. Usually, the softwares for graphing curves and surfaces come equipped with a given

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set of thickness for the curves and it is not possible to prescribe the thickness of the curve at will. Moreover, if the space curves are represented as tubes, this helps also the intuition, it is possible to use different effects (shadow, transparency, etc.). It would be possible even to use the tubular surface for graphing the wireframe of a surface. There is, of course, a price that we have to pay: the speed. It is, clearly, more expensive to represent a curve as a tubular surfaces than to representing using solids. However, this problem can be partially solved using spline approximations. Moreover, this is really a problem only for the interactive visualization and it should be always possible to use the "tube approach" only when the "still" graphics is produces.

The aim of this paper is to discuss propose new approaches to the the tubular and canal surfaces (which, as we shall see, are natural generalization of tubular surfaces), providing explicit representation of them as *swept surfaces* as well as parameterization of the surfaces obtained by their transformation by inversion. Finally, we suggest some possible application of tubular surfaces in scientific vizualisation (for differential geometry and topology).

We mention that all the figures from the paper have been realized by using the C++ graphical library Open Geometry, based on OpenGL, elaborated at the Technical University of Wien by G. Glaeser and H. Stachel (see [8] for a recent survey).

2. Tubular surfaces

We shall give here the mathematical description of tubular surfaces associated to space curves. For all the geometrical notions, see [3].

Definition 2.1. Let $\mathbf{r} : I \to \mathbb{R}^3$ be a smooth, regular space curve. The *tubular surface* associated to $\mathbf{r}$, of radius a, is, by definition, the envelope of the family of spheres of radius a, with the center on the curve (see figure 2).

Remark 2.1. Clearly, if $\mathbf{r}$ is a straight line, then the tubular surface of radius a associated to it is just the circular cylinder of radius a, having $\mathbf{r}$ as symmetry axis. If, on the other hand, $\mathbf{r}$ is a circle, then the corresponding tubular surface is a *torus*.

We shall assume, hereafter that $\mathbf{r}$ is *biregular*, in other words, along the curve the following condition is fulfilled: $\mathbf{r}'(t) \times \mathbf{r}''(t) \neq 0$. We notice that this condition do not hold for straight lines. Then, to each point of the curve, we can associate the *Frenet frame*, i.e. the frame $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$ formed by the following three vectors (see [3]):

(i) $\boldsymbol{\tau} = \dfrac{\mathbf{r}'}{\|\mathbf{r}'\|}$ – the unit tangent vector;

(ii) $\boldsymbol{\beta} = \dfrac{\mathbf{r}' \times \mathbf{r}''}{\|\mathbf{r}' \times \mathbf{r}''\|}$ – the unit binormal vector;

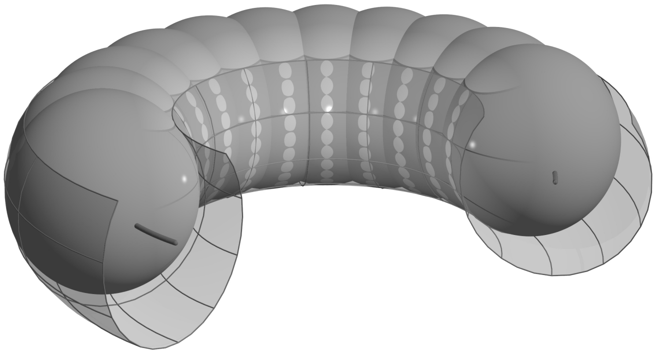


FIGURE 1. Tubular surfaces as envelopes of spheres

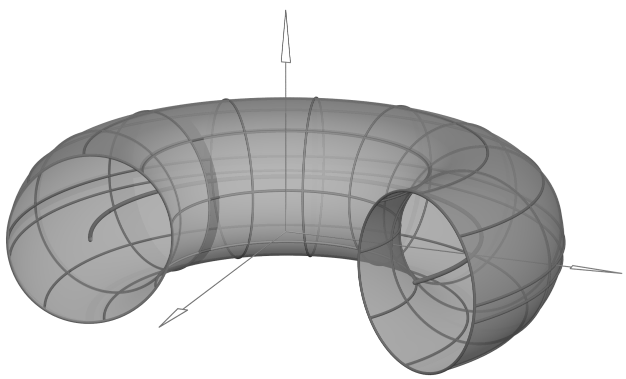


FIGURE 2. The torus as a tubular surface

(iii) $\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}$ – the unit principal normal vector.

Now, it is easy to see that the contact between the spheres from the family and the tubular surface is a great circle of the sphere, lying in the *normal plane* of the generating curve. Let us describe, then, a very simple method of parameterization of the tubular surface. Take the parameter along the generating curve to be one of the parameters and denote by $\mathbf{R}$ the position vector of a point on the surface. As we mentioned earlier, this point lies in the normal plan to the generating curve at a point t. On the other hand, it lies on a circle of radius a, situated in this plan, with the center at the point $\mathbf{r}(t)$ from the curve. Let us denote by $\boldsymbol{\rho}$ the vector connecting the point from the curve with the point from the surface. Then, clearly, we have

$$\mathbf{R} = \mathbf{r}(t) + \boldsymbol{\rho}. \tag{1}$$

The vector $\boldsymbol{\rho}$ itself lies in the normal plane. Let us denote by θ the angle between the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\nu}$. Then, as one can see immediately, we have

$$\boldsymbol{\rho} = a(\boldsymbol{\nu}(t)\cos\theta + \boldsymbol{\beta}(t)\sin\theta). \tag{2}$$

Combining (1) and (2), we see that we obtained a parameterization of the tubular surface,

$$\mathbf{R}(t,\theta) = \mathbf{r}(t) + a\boldsymbol{\nu}(t)\cos\theta + a\boldsymbol{\beta}(t)\sin\theta. \tag{3}$$

Implementation issues. When using a computer algebra system, usually we can compute, formally, the derivatives of functions. However, generally, these facilities are not available. Apparently, we are left with the problem of evaluating numerically the derivatives in order to construct the Frenet frame. However, in practice, we usually do not follow this method. The solution is quite simple, if we use the geometrical interpretations of the tangent line and the osculating plane (the plane containing the tangent versor and the principal normal versor). Namely (see [3])

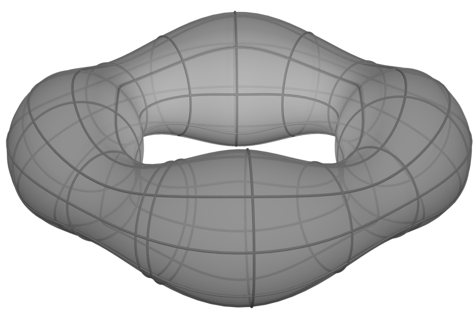
(1) the tangent line at a point of a curve is the limit position of the straight line determined by the given point and a neighboring point of the curve, when this one approaches the given point;
(2) the osculating plane at a point of the curve is the limit position of a plane determined by the given point and two neighboring points of the curve, when these ones approaches the given point.

Thus, the algorithm for finding the Frenet at a point of the curve goes like that:

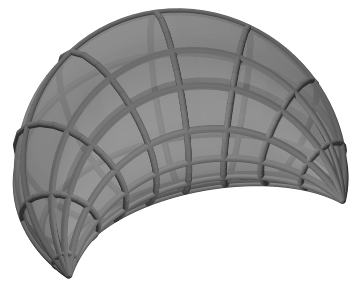
(1) Choose another point of the curve, which is close enough to the given one, by varying a little bit the parameter. These two points determine a straight line, which, within a precision limit which should be prescribed by the user, approximates the tangent line. We take then two points of the line. They determine a vector which, after normalization, can be considered as an approximation of $\boldsymbol{\tau}$ (the tangent versor)
(2) Choose yet another point of the curve, beside the one already chosen at the previous step, again, close enough to the initial point. The three points will determine a plane, which is an approximation of the osculating plane. The unit normal vector of this plane will be, the, an approximation of the binormal vector, $\boldsymbol{\beta}$.
(3) $\boldsymbol{\nu}$ (or, rather, its approximation) will be obtained by taking the cross product of the vectors constructed previously.

3. Canal surfaces

The tubular surfaces are particular cases of more general surfaces, called *canal surfaces*, which are envelopes of family of spheres, of *variable* radius, with the center on a given curve. It is very easy to see that they have a parameterization of the form (3), only that now a is not any longer a constant. For instance, in the figure 3(a) we represented the canal surfaces obtained as the envelope of the spheres of radius $4 + \sin(2t)$, with the centers on the circle $\mathbf{r}(t) = (10\cos t, 10\sin t, 0)$, while in the figure 3(b) we represented a member of a family of surfaces which are very important for computer graphics, a *Dupin cyclide*. The canal surfaces (and, in particular, the tubular surfaces) have the property that they have a family of curvature lines which are circles (the contacts between the surfaces and the generating spheres). The Dupin cyclides have the property that *all* the curvature lines are circles and it is exactly this the reason why they are so useful in computer graphics, for blending more complex surfaces (see, for instance, the review paper by Boehm ([4]).



(a) A canal surface on a circle



(b) A Dupin cyclide

FIGURE 3. Canal surfaces

4. Canal surfaces as swept surfaces

Swept surfaces (see, for instance, [10]) are, roughly speaking, surfaces obtained by "sweeping" a curve along another curve. More specifically, a swept surface is a surface obtained in the following way. We denote by $\mathbf{r}(t)$ the trajectory curve and by $\mathbf{C}(\theta)$ the section curve. Then a swept surface determined by the two curves is a surface of the form

$$\mathbf{S}(t,\theta) = \mathbf{r}(t) + M(t) \cdot \mathbf{C}(\theta), \tag{4}$$

where $M(t)$ is a 3×3 matrix, describing the transformations applied to the curve $\mathbf{C}$ along the trajectory. More precisely, M describes the rotations and the scalings applied to the section curve. In practice, usually both $\mathbf{T}$ and $\mathbf{C}$ are given as NURBS curves and the intention is to represent the surface (itself as a NURBS surface. Usually, for an arbitrary transformation matrix, this might not be possible, therefore we have to use approximations. In fact, only in the very particular case of *translation surfaces*, when the matrix M is the identity we have an exact

NURBS representation for the surface, if the two generating curves are given as NURBS curves. However, this case is not very interesting, as one can see immediately that the only translation surface that is also a tubular surface is the circular cylinder and, generally speaking, the translation surfaces which are canal surfaces are surfaces of revolution.

We shall deduce now the precise form of the equation of a tubular surface as a swept surface, in the sense that we shall indicate the form of the matrix M for the case of tubular surfaces. First of all, the curve $\mathcal{C}$ is a circle of radius a, which we assume to be situated in the xOy plane, in other words, its equation is of the form

$$\mathcal{C}(\theta) = (a\cos\theta, a\sin\theta, 0). \tag{5}$$

What we have to do now is to rotate the curve $\mathbf{C}$ in such a way that, after rotation, it will lye in the normal plane of the curve $\mathbf{r}$. Thus, for any t, the transformation matrix M should turn the xOy plane into the normal plane of the curve at the point $\mathbf{r}(t)$. The idea, of course, is to find a three-dimensional rotation that will turn the axes of the coordinate system (translated at the point of the curve!) into the axes of the Frenet frame, in such a way that the z-axis correspond to the tangent of $\mathbf{r}$, the x-axis – to the principal normal and the y-axis – to the binormal. But then (see [7]), we know that the columns of the rotation matrix should be nothing but the versors of the new direction. Thus, in this case, we have

$$M(t) = \begin{bmatrix} \boldsymbol{\nu}(t) & \boldsymbol{\beta}(t) & \boldsymbol{\tau}(t) \end{bmatrix}. \tag{6}$$

More generally, if we consider an arbitrary canal surface, instead of a tubular one, then an uniform scaling is, also, involved and then the transformation matrix will be the product of matrices:

$$M(t) = [\boldsymbol{\nu}(t)\ \boldsymbol{\beta}(t)\ \boldsymbol{\tau}(t)] \cdot \begin{bmatrix} f(t) & 0 & 0 \\ 0 & f(t) & 0 \\ 0 & 0 & f(t) \end{bmatrix} \equiv [f(t)\cdot\boldsymbol{\nu}(t)\ f(t)\cdot\boldsymbol{\beta}(t)\ f(t)\cdot\boldsymbol{\tau}(t)], \tag{7}$$

where $f(t)$ is the scaling factor.

Remark 4.1. It is very easy to check that, indeed, the equation (4), with the transformation matrix (6) and the sweeping curve (5) coincide with the equation of the tubular surface established earlier, namely the equation (3).

As we mentioned earlier, generally speaking, the swept surfaces are not, as such, NURBS surfaces, even if the generating curves are NURBS curves. The problem is due to the, generally complicated, structure of the transformation matrix (in particular, that of the rotation matrix). This claim is true, in particular, also for canal surfaces. There are several ways to approximate swept surfaces by NURBS. They are described, in details, in [10].

5. The inversion of tubular surfaces and canal surfaces

A geometrical transformation that it was not exploited yet properly in computer graphics is the *inversion*. We recall, (see, for instance [2]) that the inversion (in space), is a geometrical transformation, defined by a point P (called the *center of the inversion*) and a real number k,(different from zero), called the *power of the inversion*. The inversion $\mathcal{I}$ is defined on $\mathbb{R} \setminus P$, with values in the entire space, such that, for each point $M \neq P$, $\mathcal{I}(P)$ is a point that lies on the straight line PM and the following relation is fulfilled:

$$\overline{P\mathcal{I}(M)} \cdot \overline{PM} = k, \tag{8}$$

where with an overline we denoted the signed length of the segments. If k is positive, M and $\mathcal{I}(M)$ are on the same halfline with respect to P and the inversion is called *positive*. Otherwise, they are on opposite sides and the inversion is called *negative*. The sphere with center at P and with radius $\sqrt{|k|}$ is called the *inversion sphere* or *inverting sphere*.

We are going to find now explicitly the equation of the inverse of a canal surface. Let P be the center of inversion and k – its power. We assume that the radius vector of the center is $\mathbf{r}_0$ and M an arbitrary point in space, different from M, with the position vector $\mathbf{r}$. We denote by M' its inverse and by $\mathbf{r}'$ the radius vector of the inverse. We intend to find this radius vector, in terms of $\mathbf{r}$ and $\mathbf{r}_0$. First of all, as the points P, M and M' are colinear, the relation (8) is equivalent to

$$\overrightarrow{PM} \cdot \overrightarrow{PM'} = k \tag{9}$$

or, which is the same,

$$(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}' - \mathbf{r}_0) = k. \tag{10}$$

On the other hand, as the three points are colinear, we also have

$$\mathbf{r}' - \mathbf{r}_0 = \lambda(\mathbf{r} - \mathbf{r}_0), \tag{11}$$

where λ is a constant to be determined. Combining (10) and (11), we get to the conclusion that

$$\mathbf{r}' = \mathbf{r}_0 + \frac{\mathbf{r} - \mathbf{r}_0}{(\mathbf{r} - \mathbf{r}_0)^2}. \tag{12}$$

Thus, the inverse of a canal surface with respect to a point of radius vector $\mathbf{r}_0$ will have the equation

$$\mathbf{R}'(t, \theta) = \mathbf{r}_0 + \frac{\mathbf{r}(t) - \mathbf{r}_0 + f(t)(\boldsymbol{\nu}(t)\cos\theta + \boldsymbol{\beta}(t)\sin\theta)}{(\mathbf{r}(t) - \mathbf{r}_0 + f(t)(\boldsymbol{\nu}(t)\cos\theta + \boldsymbol{\beta}(t)\sin\theta))^2}, \tag{13}$$

where $f(t)$ is the scaling function. Of course, if f is a constant, we get the equation of the inverse of tubular surface. The transformation by inversion should play a more important role in computer graphics, due to several appealing qualities (see, for instance, ([2]) for a discussion of the properties of inversion):

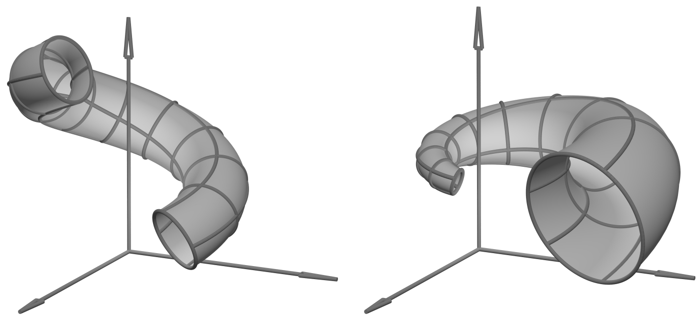


FIGURE 4. A tubular surface and its inverse

- it is a conformal transformation (leaves invariant the angles);
- leaves invariant the curvature lines of surfaces.

Also, a very important property is that the circles that do not pass through the inversion center are also transformed into circles. As a direct consequence of these properties, one can see easily that a torus is transformed into a Dupin cyclide (these are the only surfaces for which both families of curvature lines are circles (as it happens, also, for the torus, which is, also, a particular case of a cyclide)

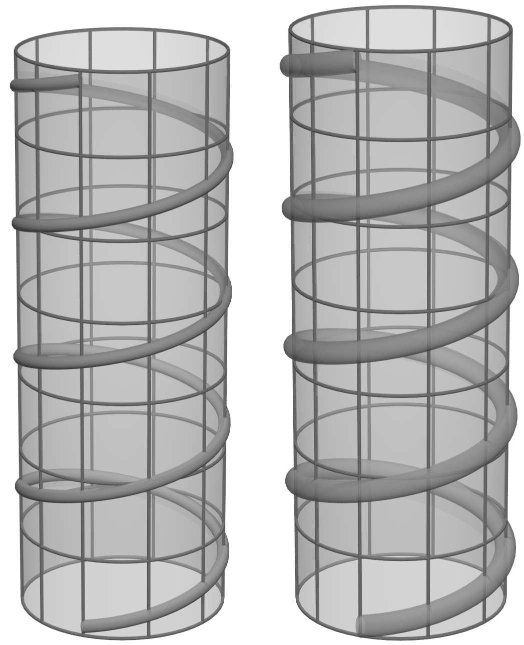
By inverting surface, with a suitable choice of the center of power of inversion, we can produce surfaces with interesting shapes, sometimes we can even imitate natural shapes. In the figure (5) we show a helical surface (obtained, for instance, as the envelope of a family of spheres of radii .5, with the center on a cylindrical helix of parameter .5, lying on cylinder of radius 1.4, having the z-axis as the symmetry axis, as well as the inverse of this surface with respect to the origin, with the power of inversion equal to 2.

6. Applications

We shall illustrate here two applications of the tubular surface for the visualisation of surfaces in differential geometry and topology.

We produced, first of all, two plots of a geodesic on a circular cylinder (see figure 6). Obviously, the method works also in more general circumstances, but we only want to emphasize the utility as such an approach, as opposed to the classical one. In the first graphic we have drawn a thinner tube, without transparency, in the second we took a bigger radius an also used transparency also for the tube, not only for the cylinder.

As a second example, we mention the illustration of the classification theorem for compact, orientable surfaces in $\mathbb{R}^3$. As it is known (see, for instance, [1]), any compact, orientable surface of genus g is homeomorphic to a sphere with g handles (see figure 6), or , which is the same, with a torus with g holes (see figure 6). The



(a) Thinner, no transparency (b) Thicker, with transparency

Figure 5. A geodesic on the cylinder

images were constructed very easily, just as unions of a sphere and three tori, respectively as a union of three tori.

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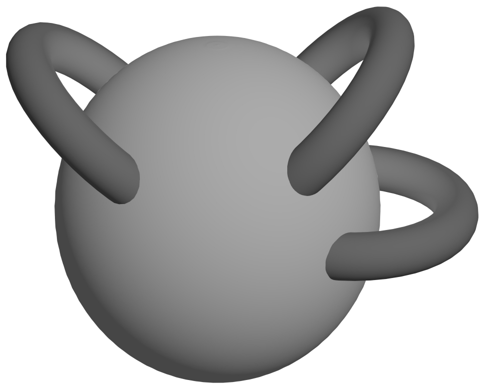

FIGURE 6. A sphere with three handles

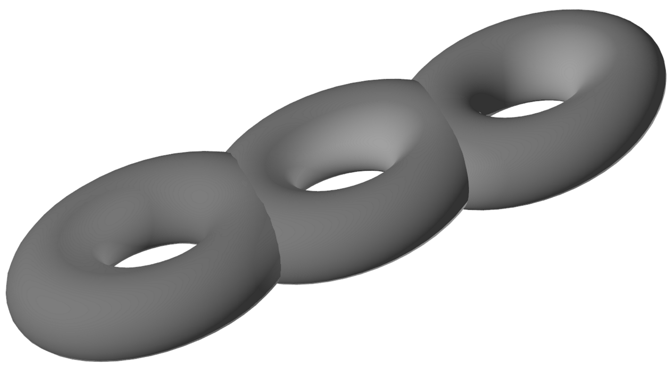

FIGURE 7. A torus with three holes

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"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 1, Kogălniceanu Street, Cluj-Napoca, Romania

E-mail address: pablaga@cs.ubbcluj.ro