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A WEIGTHED-PATH-FOLLOWING METHOD FOR THE LINEAR COMPLEMENTARITY PROBLEM

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Abstract. In a recent paper [3] a weighted path-following interior point method (IPM) has been developed to solve linear programs (LP) based on a method for finding a new family of search directions. In this paper we describe a similar approach for linear complementarity problems (LCP). We prove that the algorithm performs the same number of iterations as in [3] .

1. Introduction

Formally, generalized path-following interior point methods (or the so-called target-following methods) are related to the classical central path methods but they are more general in the sense that the barrier parameter is a multidimensional vector and not a real number. Geometrically, these methods are based on the observation that with every algorithm which follows the central path we can associate a target sequence on this central path. A good survey of this concept can be found in [9] . Weighted-path following methods can be seen as a particular case of target-following methods. These methods were studied by Ding and Li [4] for primal-dual linear complementarity problems. Recently, Darvay [3] has been developed a weighted path-following algorithm for solving the linear optimization (LO) problem, based on a new method for finding a new family of search directions. In this paper, we describe a similar approach for solving the linear complementarity problem (LCP).

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This paper is organized as follows. In the next section the LCP problem, the statement of the problem and the associated weighted problem are presented. Section 3 deals with the existence and uniqueness of the solution of the weighted problem and its differentiability. Section 4 is devoted to the search direction and the description of the algorithm. Section 5 presents the convergence analysis of the algorithm and its polynomial complexity. Section 6 ends the paper with a conclusion.

1.1. **Notations.** Our notation is the usual one. In particular, $\Re^n$ denotes the space of real $n-$dimensional vectors and $\Re^n_+$ the nonnegative orthant of $\Re^n$. Let $u,\ v \in \Re^n,\ u^T v$ is their inner product, $\| u\|$ is the Euclidean norm and $\| u\|_\infty$ is maximum norm. Given a vector u in $\Re^n$, $U =$diag(u) is the $n \times n$ diagonal matrix with $U_{ii} = u_i$ for all i, where U_{ii} denotes the i-th element on the diagonal of U. The vector $e = [1, \dots, 1]^T$ is the vector of ones in $\Re^n$. Given the vectors x and y in $\Re^n$, $xy = [x_1y_1, x_2y_2, \dots, x_ny_n]^T$ denotes the coordinate-wise product of x and y, and $\langle x, y\rangle$ the scalar product of x and y. We shall use also the notation $\frac{x}{y} = \left[\frac{x_1}{y_1}, \frac{x_2}{y_2}, \dots, \frac{x_n}{y_n}\right]^T$ with $y_i \neq 0$ for all i. For a given arbitrary function ψ, and an arbitrary vector x we will use the notation $\psi(x) = [\psi(x_1), \dots, \psi(x_n)]^T$. $d(A, B)$ is the distance between the sets A and B.

2. The LCP problem and the statement of the problem

The linear complementarity problem (LCP) is defined as:
find $x \geqslant 0$ and $y \geqslant 0$ such that

$$y = Mx + q,\ x^T y = 0 \tag{1}$$

where M is a given $(n \times n)$ real matrix and q is a given $n-$dimensional real vector, the inequalities are understood to be components-wise. The complementarity condition $x^T y = 0$, is equivalent to $x_iy_i = 0$, for $i = 1, 2, ..., n$.

The linear complementarity problems model many important mathematical problems. The books [1, 7] are good documentations of complementarity problems.

The feasible set, the strict feasible set and the solution set of the problem (1) are denoted respectively by

$$\begin{aligned}\mathcal{F} &= \left\{(x,y)\in\mathcal{R}^2 : Mx+q=y,\ x\geqslant 0,\ y\geqslant 0\right\},\\ \mathcal{F}^0 &= \left\{(x,y)\in\mathcal{F} :\ x>0,\ y>0\right\},\end{aligned}$$

and

$$\mathcal{S}_{cp}=\left\{(x,y)\in\mathcal{F} : x_iy_i=0, i=1,2,...,n\right\}.$$

Throughout the paper we make the following assumptions.
Assumption 1 $\mathcal{F}^0\neq\emptyset$.
Assumption 2 M is a positive semidefinite matrix.
Assumption 1 implies that $\mathcal{F}^0$ is the relative interior of $\mathcal{F}$ and also that the set of solution of (LCP) is nonempty convex and compact.
We formulate (1) into the equivalent convex minimization problem:

$$\min\left[x^Ty \ \text{ s.t. } x\geqslant 0,\ y\geqslant 0, Mx+q=y\right]. \tag{2}$$

We observe, that if (x,y) is a solution of the (LCP), then the global minimizer of (2) is zero, see [13]. We associate with (2) the following weighted problem:

$$(\mathrm{P}_w)\qquad \min\left[g_w(x,y)=x^Ty-\sum_{i=1}^{n}w_i^2\ln\frac{x_iy_i}{w_i^2}\right],\qquad \text{s.t. } (x,y)\in\mathcal{F}^0.$$

where $w^2=\left[w_1^2,w_2^2,\ldots,w_n^2\right]^T$ for a given positive vector w.
Denote by $\mathcal{L}(x,y,z,w)$, the Lagrangian of the problem (P_w)

$$\mathcal{L}(x,y,z,w)=x^Ty+\sum_{i=1}^{n}w_i^2\ln\left(\frac{w_i^2}{x_iy_i}\right)-z^T(Mx+q-y). \tag{3}$$

The first order optimality conditions for (3) give the system of nonlinear equations

$$y-X^{-1}w^2-M^Tz = 0, \tag{4}$$
$$x-Y^{-1}w^2+z = 0, \tag{5}$$
$$Mx+q-y = 0, \tag{6}$$

where $z\in\mathfrak{R}^n, X=\text{diag}\,(x)$ and $Y=\text{diag}(y)$.
From the first and the second equation in $(4-6)$, we obtain

$$\begin{cases} XYe-w^2-XM^Tz = 0,\\ XYe-w^2+Yz = 0.\end{cases}$$

It follows that

$$(M^T + X^{-1}Y)z \; = \; 0. \tag{7}$$

Recall that M is positive semidefinite. Hence, for all $h \neq 0$

$$\langle \left(M^T + X^{-1}Y\right) h, h\rangle = \langle Mh, h\rangle + \langle X^{-1}Yh, h\rangle > \langle Mh, h\rangle \geqslant 0.$$

It follows that $\left(M^T + X^{-1}Y\right)$ is invertible. Thus $z = 0$, and the system (4-6) reduces to

$$\mathfrak{F}(x, y, w) = \begin{pmatrix} XYe - w^2 \\ Mx + q - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{8}$$

The system (8) can be written as

$$Mx + q = y, \quad xy = w^2, \tag{9}$$

where $xy = [x_1y_1, x_2y_{2,\cdots}, x_ny_n]^T$. This form is convenient for the analysis of the convergence of the suggested algorithm. Hence, solving the problem (P_w) is equivalent to solving the system (9).

Our next aim is to show that the weighted problem (P_w) has one unique minimizer. To this end, we follow the technique of proof from [2, 10] for the classical barrier logarithmic methods to show the existence of the minimizer.

3. Existence and the uniqueness of the minimizer of (P_w)

Let us define the following function $\tilde{g}_w$ by

$$\tilde{g}_w(x) = g_w(x, Mx + q).$$

Proposition 3.1. *Under assumptions **1** and **2**, we have*
1) the function $\tilde{g}_w$ is strictly convex on

$$\{(x, y) : x > 0,\ y = Mx + q, y > 0\}.$$

2) the problem (P_w) and the system (8) are equivalent for $w > 0$.
3) for each $w \in (0, +\infty)^n$, the problem (P_w) has one unique global minimizer or equivalently the system (8) or (9) has a unique solution denoted by $(x(w), y(w))$ with $x(w) > 0$ and $y(w) > 0$.

Proof. To prove the strict convexity of $\tilde{g}_w$, we omit first the constant part $\sum_{i=1}^{n} w_i^2 \ln w_i^2$ from it. Then we have

$$\tilde{g}_w(x) = x^T(Mx+q) - \sum_{i=1}^{n} w_i^2 \ln x_i - \sum_{i=1}^{n} w_i^2 \ln(Mx+q)_i,$$

since

$$\nabla \tilde{g}_w(x) = (M+M^T)x + q - X^{-1}w^2 - M^T\left[\operatorname{diag}(Mx+q)\right]^{-1} w^2,$$

and

$$\nabla^2 \tilde{g}_w(x) = M + M^T + X^{-2}W + M^T\left[\operatorname{diag}\left(Mx+q\right)\right]^{-2} W^2 M$$

where $W^2 = \operatorname{diag}(w^2)$.
Hence $\nabla^2 \tilde{g}_w$ is a positive definite matrix and thereby $\tilde{g}_w$ is strictly convex function on $\{(x,y) : x > 0, y = Mx+q,\ y > 0\}$.
For the second statement, we have the objective function g_w of the problem (P$_w$) can be written as

$$\sum_{i=1}^{n} \left[x_i y_i - w_i^2 \ln \frac{x_i y_i}{w_i^2}\right]$$

and since each term in brackets attains the minimum under the condition $(x,y) \in \mathcal{F}^0$, if and only if, $x_i y_i = w_i^2$, then any minimizer of (P$_w$) is a solution of the system (8) and vice versa.
For the last statement, let $w \in (0,+\infty)^n$ be fixed. We have

$$\hat{g}_w(x) = g_w(x, Mx+q) = x^T(Mx+q) + \sum_{i=1}^{n} w_i^2 \ln \frac{x_i(Mx+q)_i}{w_i^2}.$$

Recall that in view of assumption 1, $\mathcal{S}_{cp}$ is nonempty and bounded. Therefore $\left\{(x,y) \in \mathcal{F} : x^T(Mx+q) \leqslant 0\right\}$ is bounded. By assumption 2, the level set $\left\{(x,y) \in \mathcal{F} : x^T(Mx+q) \leqslant t\right\}$ is bounded for any t. It follows that the level set

$$\Omega(t) = \{x \in \Re^n :\ g_w(x, Mx+q) \leqslant t,\ Mx+q > 0\}$$

is also bounded since $\hat{g}_w$ differs only by the term

$$-\sum_{i=1}^{n} w_i^2 \ln \frac{x_i(Mx+q)_i}{w_i^2}$$

from the quadratic function $x^T(Mx+q)$. Again in view of assumption 1, it follows that $\Omega(t) \neq \emptyset$ for sufficiently large t. Finally, if $(x,y) \in \mathcal{F}^0$ approaches the boundary of $\mathcal{F}$, then

$$\ln \frac{x_i(Mx+q)_i}{w_i^2} \to +\infty.$$

This implies that $g_w(x, Mx+q) \to +\infty$. Thus $\hat{g}_w$ must have a global minimizer in $\mathcal{F}^0$denoted by $x(w)$. Since $\hat{g}_w$ is strictly convex, then $x(w)$ is unique.
If one of the components of $x(w)$ is zero, then $g_w(x, Mx+q) \to +\infty$. It follows that $x(w) > 0$. From the system (8), $y(w)$ is also determined uniquely and $y(w) > 0$. ■
The next objective is to show the differentiability of the solutions of the weighted problem (P_w), $w \mapsto x(w)$ and $w \mapsto y(w)$ on $(0,\infty)^n$.

Theorem 3.2. *The functions $w \mapsto x(w)$ and $w \mapsto y(w)$ are $\mathcal{C}^\infty$ on $(0,\infty)^n$.*

Proof. Let us recall again the mapping defined in (8) by

$$\begin{aligned} \mathfrak{F} \quad &: \quad \mathfrak{R}_+^{2n} \times \mathfrak{R}_+^n \to \mathfrak{R}^{2n} \\ \mathfrak{F}(u,w) \quad &= \quad (Mx+q-y, Xy-w^2). \end{aligned}$$

The Fréchet derivative of $\mathfrak{F}$ with respect to $u=(x,y)$ is:

$$\mathfrak{F}'_u(u,w) = \begin{pmatrix} Y & X \\ M & -I \end{pmatrix}.$$

Since

$$\begin{pmatrix} I & 0 \\ -MY^{-1} & I \end{pmatrix} \begin{pmatrix} Y & X \\ M & -I \end{pmatrix} = \begin{pmatrix} Y & X \\ 0 & -(I+MY^{-1}X) \end{pmatrix},$$

it follows that the Jacobian matrix $\mathfrak{F}'_u(u,w)$ is invertible for (u,w) with $x>0, y>0$ and $w>0$.
Let now $\bar{w}$ be fixed in $(0,+\infty)^n$ and $u(\bar{w}) = 0$. Hence $\mathfrak{F}(\bar{u},\bar{w}) = 0$. Since $\mathfrak{F}$ is continuously differentiable and $\mathfrak{F}'(u,w)$ is invertible, applying the implicit function theorem we obtain that there exists θ continuously differentiable function on a neighborhood of $\bar{w}$ such that $\mathfrak{F}(\theta(w),w) = 0$. Since the nonlinear system (8) characterizes the optimal solution $(x(w),y(w))$ of (P_w), therefore $\theta(w) = u(w)$. The function u is differentiable. By an immediate induction it is C^∞. ■

Proposition 3.3. *Let $(x(w),\ y(w))$ be a solution of (P_w). Then*
1) $x^T(w)y(w) \to 0$ as $w \mapsto 0$.
2) $d(\mathcal{S}_\varepsilon, \mathcal{S}_{cp}) \to 0$ as $\varepsilon \mapsto 0$ where

$$\mathcal{S}_\varepsilon = \left\{(x,y) : x \geqslant 0,\ y \geqslant 0,\ Mx + q = y,\ x^T y \leqslant \varepsilon\right\}.$$

3) combining 1) and 2) we have $d((x(w), y(w)), \mathcal{S}_{cp}) \to 0$ as $w \to 0$.

Theorem 3.4. *For any $w > 0$, there is a unique solution $(x(w), y(w))$ to (8) and the path $\{(x(w), y(w)) : w > 0\}$ is smooth.*

Remark 3.1. *The central path method corresponds to the path $\left\{(x(\mu e),\ y(\mu e)) :\ \mu > 0\right\}$. So, our method can be viewed as a generalization of the classical central path method.*

In the next section we describe a similar approach as in [3] to solve the (LCP). This section follows closely the argument developed in [3] for finding a new family of search directions by using the system (9).

4. New search directions and the algorithm

Let $\Re^+ = \{x \in \Re\ : x \geqslant 0\}$, and consider the function

$$\varphi \in \mathcal{C}^1,\ \varphi\ :\ \Re^+ \to \Re^+.$$

We suppose that φ is a one to one function, i.e. φ^{-1} exists. Then the system (9) can be written in the following equivalent form

$$\begin{aligned} Mx + q = y, \quad x \geqslant 0,\ y \geqslant 0, \\ \varphi(xy) = \varphi(w^2). \end{aligned} \tag{10}$$

Suppose that we have $(x, y) \in \mathcal{F}^0$, i.e. x and y are strictly feasible. Applying Newton's method for the system (10) we obtain the new class of search directions

$$\begin{aligned} M\Delta x = \Delta y, \\ y\varphi^{'}(xy)\Delta x + x\varphi^{'}(xy)\Delta y = \varphi(w^2) - \varphi(xy). \end{aligned} \tag{11}$$

Now the following notations are useful for studying the complexity of the proposed algorithm:

$$v = \sqrt{xy} \quad \text{and} \quad d = \sqrt{xy^{-1}}.$$

Observe that these notations lead to

$$d^{-1}x = dy = v. \tag{12}$$

Denote

$$d_x = d^{-1}\Delta x, \quad d_y = d\Delta y,$$

and hence, we have

$$v(d_x + d_y) = y\Delta x + x\Delta y, \tag{13}$$

and

$$d_x d_y = \Delta x \Delta y. \tag{14}$$

So the system (11) becomes

$$\begin{aligned} \bar{M} d_x &= d_y, \\ d_x + d_y &= p_v, \end{aligned}$$

where $\bar{M} = DMD$, with $D = \mathrm{diag}(d)$ and

$$p_v = \frac{\varphi(w^2) - \varphi(xy)}{v\varphi'(v^2)}.$$

As in [3] , we put $\varphi(t) = \sqrt{t}$. Hence the Newton's direction in (11) is

$$\begin{gathered} M\Delta x = \Delta y, \\ \sqrt{\frac{y}{x}}\Delta x + \sqrt{\frac{x}{y}}\Delta y = 2(w - \sqrt{xy}), \end{gathered} \tag{15}$$

with

$$p_v = 2(w - \sqrt{xy}) = 2(w - v). \tag{16}$$

We define for any vector v the following proximity measure by

$$\sigma(v, w) = \frac{\|p_v\|}{2\min(w)} = \frac{\|v - w\|}{\min(w)}, \tag{17}$$

where $\|.\|$ is the Euclidean norm and $\min(w) = \min\{w_i : 1 \leqslant i \leqslant n\}$. Now, for measuring the closeness of w^2 to the central path, we use the following quantity

$$\sigma_c(w) = \frac{\max(w^2)}{\min(w^2)},$$

where $\max(w) = \max\{w_i : 1 \leqslant i \leqslant n\}$. Now the primal-dual algorithm can be defined formally as follows.

Algorithm 4.1. We assume that $(x^0, y^0) \in \mathcal{F}^0$, and let $w^0 = \sqrt{x^0 y^0}$. Let $\epsilon > 0$ be the given tolerance, and $0 < \theta < 1$ the update parameter (default $\theta = 1/(\sigma_c(w^0)n)^{1/5}$).

begin
$x := x^0; y := y^0$
$w := w^0;$
while $x^T y > \epsilon$ do **begin**
$w := (1-\theta)w;$
compute $(\Delta x, \Delta y)$ from (15)
$x := x + \Delta x;$
$y := y + \Delta y;$
end
end.

In the following section we prove that the algorithm converges to a solution of the (LCP) in polynomial time.

5. The convergence analysis and the complexity analysis

Let

$$q_v = d_x - d_y.$$

Then

$$d_x d_y = \frac{p_v^2 - q_v^2}{4},$$

and

$$\|q_v\| \leqslant \|p_v\|. \tag{18}$$

This last result follows directly from the equality

$$\|p_v\|^2 = \|q_v\|^2 + 4d_x^T d_y.$$

In the following Lemma we give a condition to ensure the feasibility of the full step Newton.
Let $x_+ = x+\Delta x$ then $y_+ = M(x+\Delta x)+q = Mx+q+M\Delta x = y+\Delta y$.

Lemma 5.1. *Let $\sigma = \sigma(u,v) < 1$. Then the full Newton step is strictly feasible, hence*

$$x_+ > 0 \text{ and } y_+ > 0.$$

Proof. For each $0 \leqslant \alpha \leqslant 1$ let $x_+(\alpha) = x + \alpha\Delta x$ and $y_+(\alpha) = y + \alpha\Delta y$. Hence

$$x_+(\alpha)y_+(\alpha) = xy + \alpha(x\Delta y + y\Delta x) + \alpha^2 \Delta x \Delta y.$$

Now, in view of (13) and (14) we have

$$x_+(\alpha)y_+(\alpha) = v^2 + \alpha v(d_x + d_y) + \alpha^2 d_x d_y.$$

In addition from (16) we have

$$v + \frac{p_v}{2} = w,$$

and thus

$$v^2 + vp_v = w^2 - \frac{p_v^2}{4}.$$

Thereby

$$\begin{aligned} x_+(\alpha)y_+(\alpha) &= (1-\alpha)v^2 + \alpha(v^2 + vp_v) + \frac{\alpha^2}{4}(p_v^2 - q_v^2) = \\ &= (1-\alpha)v^2 + \alpha(w^2 - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}), \end{aligned} \tag{19}$$

thus the inequality $x_+(\alpha)y_+(\alpha) > 0$ holds if

$$\left\| (1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4} \right\|_\infty < \min(w^2).$$

Using (17) and (18) we get

$$\begin{aligned} &\left\| (1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4} \right\|_\infty \leqslant (1-\alpha)\left\| \frac{p_v^2}{4} \right\|_\infty + \alpha\left\| \frac{q_v^2}{4} \right\|_\infty \leqslant \\ &\leqslant (1-\alpha)\frac{\|p_v\|^2}{4} + \alpha\frac{\|q_v\|^2}{4} \leqslant \frac{\|p_v\|^2}{4} = \sigma^2 \min(w^2) < \min(w^2). \end{aligned}$$

Hence, $x_+(\alpha)y_+(\alpha) > 0$ for each $0 \leqslant \alpha \leqslant 1$. Since $x_+(\alpha)$ and $y_+(\alpha)$ are linear functions of α, then they don't change sign on the interval $[0,1]$ and for $\alpha = 0$ we have $x_+(0) > 0$ and $y_+(0) > 0$. This leads to $x_+(1) > 0$ and $y_+(1) > 0$. ■

In the next Lemma we show that $\sigma < 1$ is sufficient for the quadratic convergence of the Newton process.

Lemma 5.2. *Let $x_+ = x + \Delta x$ and $y_+ = y + \Delta y$ be the iterates obtained after a full Newton step with $v = \sqrt{xy}$ and $v_+ = \sqrt{x_+y_+}$. Suppose $\sigma = \sigma(v,w) < 1$. Then*

$$\sigma(v_+, w) \leqslant \frac{\sigma^2}{1 + \sqrt{1-\sigma^2}}.$$

Thus $\sigma(v_+, w) < \sigma^2$, which means quadratic convergence of the Newton step.

Proof. By substituting $\alpha = 1$ in (19) we have

$$v_+^2 = w^2 - \frac{q_v^2}{4}. \tag{20}$$

Using (13) and (20) we obtain

$$\begin{aligned}\min(v_+^2) &\geqslant \min(w^2) - \frac{\|q_v^2\|_\infty}{4} \geqslant \min(w^2) - \frac{\|q_v\|^2}{4} \geqslant \\ &\geqslant \min(w^2) - \frac{\|p_v\|^2}{4} = \min(w^2)(1-\sigma^2),\end{aligned}$$

and this relation yields

$$\min(v_+) \geqslant \min(w)(\sqrt{1-\sigma^2}). \tag{21}$$

Furthermore, from (17) and (21) we get

$$\begin{aligned}\sigma(v_+, w) &= \frac{\|w - v_+\|}{\min(w)} = \frac{1}{\min(w)} \left\| \frac{w^2 - v^2}{w + v^+} \right\| \leqslant \\ &\leqslant \frac{\left\|w^2 - v_+^2\right\|}{\min(w)(\min(w) + \min(v_+))} \leqslant \frac{\|p_v^2\|}{(2\min(w))^2(1-\sqrt{1-\sigma^2})} \\ &\leqslant \frac{1}{1-\sqrt{1-\sigma^2}} \left(\frac{\|p_v\|}{2\min(v)} \right)^2 = \frac{\sigma^2}{1-\sqrt{1-\sigma^2}}.\end{aligned}$$

This proves the lemma. ■ In the following lemma we find an upper bound for the duality gap obtained after a full Newton step.

Lemma 5.3. *Let $x_+ = x + \Delta x$ and $y_+ = y + \Delta y$. Then the duality gap is*

$$x_+^T y_+ = \left\|w^2\right\| - \frac{\|q_v\|^2}{4},$$

hence

$$x_+^T y_+ \leqslant \left\|w^2\right\|.$$

Proof. In view of (19) and with $\alpha = 1$, we have

$$x_+ y_+ = w^2 - \frac{q_v^2}{4},$$

then

$$x_+^T y_+ = e^T(x_+ y_+) = e^T w^2 - \frac{e^T q_v^2}{4} = \left\|w^2\right\| - \frac{\|q_v\|^2}{4}.$$

Hence

$$x_+^T y_+ \leqslant \left\| w^2 \right\|.$$

The proof is complete. ■

The next Lemma discusses the influence on the proximity measure of the Newton process followed by a step along the weighted path.

Lemma 5.4. [3] *Let* $\sigma = \sigma(v, w) < 1$ *and* $w_+ = (1-\theta)w$, *where* $0 < \theta < 1$. *Then*

$$\sigma(v_+, w_+) \leqslant \frac{\theta}{1-\theta}\sqrt{\sigma_c(w)n} + \frac{1}{1-\theta}\sigma(v_+, w).$$

Furthermore, if $\sigma \leqslant 1/2$, $\theta = 1/(\sigma_c(w)n)^{1/5}$ *and* $n \geqslant 4$ *then we get* $\sigma(v_+, w_+) \leqslant 1/2$.

Lemma 5.5. [3] *Assume that* x^0 *and* y^0 *are strictly feasible, and let* $w^0 = \sqrt{x^0 y^0}$. *Moreover, let* x^k *and* y^k *be the vectors obtained after* k *iterations. Then the inequality* $\left(x^k\right)^T y^k \leqslant \epsilon$ *is satisfied for*

$$k \geqslant \left\lceil \frac{1}{2\theta} \ln \frac{(x^0)^T y^0}{\epsilon} \right\rceil.$$

For the default $\theta = 1/(\sigma_c(w^0)n)^{1/5}$ we obtain the following theorem.

Theorem 5.6. [3] *Suppose that the pair* $(x^0, y^0) \in \mathcal{F}^0$, *and let* $w^0 = \sqrt{x^0 y^0}$. *If* $\theta = 1/(\sigma_c(w^0)n)^{1/5}$ *then Algorithm 4.1 requires at most*

$$\left\lceil \frac{5}{2}\sqrt{\sigma_c(w^0)n} \ln \frac{(x^0)^T y^0}{\epsilon} \right\rceil$$

iterations. For the resulting vectors we have $\left(x^k\right)^T y^k \leqslant \epsilon$.

6. Conclusion

In this paper, we have described a similar approach to the one developed by Darvay for linear programs, to solve the monotone linear complementarity problem. Here we have transformed the (LCP) problem into an equivalent convex minimization problem based on some weighted methods. We have adopted the function developed in [3] to obtain new search directions and to develop the new primal-dual algorithm. We have proved that this algorithm performs no more than

$$\left\lceil \frac{5}{2}\sqrt{\sigma_c(w^0)n} \ln \frac{(x^0)^T y^0}{\epsilon} \right\rceil$$

iterations. This result is the same as the one found for the algorithm developed for the (LP).

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