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A NEW ALGORITHM FOR SOLVING SELF-DUAL LINEAR OPTIMIZATION PROBLEMS

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Abstract. Recently in [3] we have defined a new method for finding search directions for interior point methods (IPMs) in linear optimization (LO). Using one particular member of the new family of search directions we have developed a new primal-dual interior point algorithm for LO. We have proved that this short-update algorithm has also the $O(\sqrt{n}\log\frac{n}{\epsilon})$ iteration bound, like the standard primal-dual interior point algorithm. In this paper we describe a similar approach for self-dual LO problems. This method provides a starting interior feasible point for LO problems. We prove that the iteration bound is $O(\sqrt{n}\log\frac{n}{\epsilon})$ in this case too.

1. Introduction

In this paper we discuss polynomial methods for LO. The first polynomial algorithm for solving LO problems is the ellipsoid method of Khachiyan [6]. This method is important from a theoretical point of view, but is not so efficient in practice. An alternative variant was defined in 1984 by Karmarkar [5]. His projective method is the first IPM for LO. The field of IPMs has been very active since 1984. For an overview of results see the following books [1, 2, 10, 13, 14]. Let us consider the LO problem in canonical form

$$(P) \qquad \begin{aligned} &min\ c^T\xi \\ s.t.\quad &A\xi \geq b, \\ &\xi \geq 0, \end{aligned}$$

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where $A \in \Re^{m\times k}$ with $rank(A) = m$, $b \in \Re^m$ and $c \in \Re^k$. The dual of this problem is:

$$(D) \qquad \begin{aligned} & max\ b^T\pi \\ s.t.\quad & A^T\pi \le c, \\ & \pi \ge 0. \end{aligned}$$

It is well-known the following theorem.

Theorem 1.1 (strong duality) *Let $\xi \ge 0$ and $\pi \ge 0$ so that $A\xi \ge b$ and $A^T\pi \le c$, in other words ξ is feasible for (P) and π for (D). Then ξ and π are optimal if and only if $c^T\xi = b^T\pi$.* ■

This theorem implies that if (P) and (D) have optimal solutions then

$$(1) \qquad \begin{aligned} A\xi - z = b, &\quad \xi \ge 0, \quad z \ge 0, \\ A^T\pi + w = c, &\quad \pi \ge 0, \quad w \ge 0, \\ b^T\pi - c^T\xi = \rho, &\quad \rho \ge 0 \end{aligned}$$

has also a solution, where $z \in \Re^m$, $w \in \Re^k$ and $\rho \in \Re$ are slack variables. Furthermore, every solution of (1) provides optimal solutions of (P) and (D). Let us introduce the matrix $\bar{M}$ and the vectors $\bar{x}$ and $\bar{s}(\bar{x})$ as

$$\bar{M} = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} \pi \\ \xi \\ \tau \end{bmatrix}, \quad \text{and} \quad \bar{s}(\bar{x}) = \begin{bmatrix} z \\ w \\ \rho \end{bmatrix},$$

where $\tau \in \Re$. Consider the following homogeneous system

$$(2) \qquad \bar{s}(\bar{x}) = \bar{M}\bar{x}, \quad \bar{x} \ge 0, \quad \bar{s}(\bar{x}) \ge 0.$$

We mention that system (2) is the so-called *Goldman-Tucker model* [4, 12]. Let $\bar{n} = m + k + 1$ and observe that the matrix $\bar{M} \in \Re^{\bar{n}\times\bar{n}}$ is skew-symmetric, i.e. $\bar{M}^T = -\bar{M}$. Now we can state the following theorem.

Theorem 1.2 *Consider the primal-dual pair (P) and (D). Then we have*

1. *If ξ and π are optimal solutions of (P) and (D) respectively, then for $\tau = 1$ and $\rho = 0$ we obtain that $\bar{x}$ is a solution of (2).*
2. *If $\bar{x}$ is a solution of (2), then we have $\tau = 0$ or $\rho = 0$, thus we cannot have $\tau\rho > 0$.*
3. *If $\bar{x}$ is a solution of (2) and $\tau > 0$, then $(\frac{\xi}{\tau}, \frac{\pi}{\tau})$ is an optimal solution of the primal-dual pair (P)-(D).*
4. *If $\bar{x}$ is a solution of (2) and $\rho > 0$, then at least one of the problems (P) and (D) are infeasible.*

Proof: The first statement follows from the strong duality theorem. To prove the second one observe that

$$0 \le \tau\rho = \tau b^T\pi - \tau c^T\xi = \pi^T(\tau b) - \tau c^T\xi = \pi^T A\xi - \pi^T z - \pi^T A\xi - \pi^T w \le 0.$$

Thus $\tau\rho = 0$, and we get $\tau = 0$ or $\rho = 0$. Using this result the third assertion follows from Theorem 1.1. To prove the last statement suppose that both problems are feasible and $\rho > 0$. Thus there exists $\hat{\xi} \ge 0$ and $\hat{\pi} \ge 0$ so that $A\hat{\xi} \ge b$ and $A^T\hat{\pi} \le c$. From $\rho > 0$ we get $\tau = 0$, therefore $A\xi \ge 0$ and $A^T\pi \le 0$. Furthermore, from $\rho > 0$ we obtain that $b^T\pi > 0$ or $c^T\xi < 0$. If $b^T\pi > 0$ then

$$0 < b^T\pi \le \hat{\xi}^T A^T\pi \le 0,$$

and if $c^T\xi < 0$ then

$$0 > c^T\xi \ge \hat{\pi}^T A\xi \ge 0,$$

hence in both cases we have a contradiction. Thus the proof is complete. ■

In the next section we shall use the system (2) to accomplish the self-dual embedding of the primal-dual LO pair.

2. Self-Dual Embedding

In this section we investigate a generalized form of the system (2). Our approach follows the method proposed in [10]. Let us consider the LO problem

$$(\overline{SP}) \qquad \begin{aligned} & \mathit{min}\ \bar{q}^T\bar{x} \\ s.t.\quad & \bar{M}\bar{x} \ge -\bar{q}, \\ & \bar{x} \ge 0, \end{aligned}$$

where $\bar{M} \in \Re^{\bar{n}\times\bar{n}}$ is a skew-symmetric matrix, $\bar{q} \in \Re^{\bar{n}}$ and $\bar{q} \ge 0$. Moreover, let

$$\bar{s}(\bar{x}) = \bar{M}\bar{x} + \bar{q}.$$

We are going to solve $(\overline{SP})$ with an IPM, thus we need starting feasible solutions, so that $\bar{x} > 0$ and $\bar{s}(\bar{x}) > 0$. We say that in this case the problem $(\overline{SP})$ satisfies the *interior point condition* (IPC). Unfortunately such starting feasible solution for the problem $(\overline{SP})$ does not exist, but we can construct another problem equivalent to $(\overline{SP})$ which satisfies the IPC. For this purpose let

$$r = e - \bar{M}e \quad \text{and} \quad n = \bar{n} + 1,$$

where e denotes the all-one vector of length $\bar{n}$. Furthermore, introduce the notations

$$M = \begin{bmatrix} \bar{M} & r \\ -r^T & 0 \end{bmatrix}, \quad x = \begin{bmatrix} \bar{x} \\ \vartheta \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 0 \\ n \end{bmatrix},$$

and consider the problem

$$(SP)\qquad \begin{aligned} &\min\ q^T x \\ s.t.\quad &Mx \geq -q, \\ &x \geq 0. \end{aligned}$$

Observe that the matrix M is also skew-symmetric, and problem (SP) satisfies the IPC. Indeed, we have

$$M \begin{bmatrix} e \\ 1 \end{bmatrix} + q = \begin{bmatrix} \bar{M} & r \\ -r^T & 0 \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ n \end{bmatrix} = \begin{bmatrix} \bar{M}e + r \\ -r^T e + n \end{bmatrix} = \begin{bmatrix} e \\ 1 \end{bmatrix}.$$

We have used that the matric $\bar{M}$ is skew-symmetric, thus $e^T \bar{M} e = 0$, and this equality yields

$$-r^T e + n = -(e - \bar{M}e)^T e + n = 1.$$

In order to solve the problem (SP) we use an IPM. Let

$$s = s(x) = Mx + q,$$

and consider the path of *analytic centers* [11], the primal-dual *central path*

$$(3)\qquad \begin{aligned} Mx + q &= s, \\ xs &= \mu e, \end{aligned}$$

where $\mu > 0$, and xs is the coordinatewise product of the vectors x and s, i.e.

$$xs = [x_1 s_1, x_2 s_2, \ldots, x_n s_n].$$

In fact for an arbitrary function f, and an arbitrary vector x we will use the notation

$$f(x) = [f(x_1), f(x_2), \ldots, f(x_n)]^T.$$

It is well-known that if the IPC holds for the problem (SP), then the system (3) has a unique solution for each $\mu > 0$. IPMs generally follow the central path by using Newton's method. In the next section we are going to formulate an equivalent form of the central path, and we shall apply Newton's method to obtain new search directions.

3. A New Class of Directions

New search directions have been studied recently by Peng, Roos and Terlaky [7, 9, 8]. In a recent paper [3] we have proposed a different approach for defining a new class of directions for LO. In this section we propose a similar approach for the self-dual problem (SP). Thus, we introduce a new class of directions for the problem (SP). Let $\Re^+ = \{x \in \Re \mid x \geq 0\}$, and let us consider the function

$$\varphi \in C^1, \qquad \varphi : \Re^+ \to \Re^+,$$

and suppose that the inverse function φ^{-1} exists. Then the system of equations which defines the central path (3) is equivalent to

$$
\begin{aligned}
Mx + q &= s, \\
\varphi\left(\frac{xs}{\mu}\right) &= \varphi(e).
\end{aligned}
\tag{4}
$$

Using Newton's method for the system (4) we obtain new search directions for the problem (SP). Denote

$$
v = \sqrt{\frac{xs}{\mu}},
$$

and assume that $(x, s) > 0$ and $Mx + q = s$, thus x is an interior feasible solution of the problem (SP). Applying Newton's method for the system (4) we get

$$
M\Delta x = \Delta s, \tag{5a}
$$

$$
\frac{s}{\mu}\varphi\prime\left(\frac{xs}{\mu}\right)\Delta x + \frac{x}{\mu}\varphi\prime\left(\frac{xs}{\mu}\right)\Delta s = \varphi(e) - \varphi\left(\frac{xs}{\mu}\right) \tag{5b}
$$

We introduce the notations

$$
d_x = \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}.
$$

We have

$$
\mu v(d_x + d_s) = s\Delta x + x\Delta s, \tag{6}
$$

and

$$
d_x d_s = \frac{\Delta x \Delta s}{\mu}. \tag{7}
$$

Consequently (5b) can be written in the following form

$$
d_x + d_s = p_v, \tag{8}
$$

where

$$
p_v = \frac{\varphi(e) - \varphi(v^2)}{v\varphi\prime(v^2)}.
$$

Now using that M is skew-symmetric we get

$$
\Delta x^T \Delta s = \Delta x^T M \Delta x = -\Delta x^T M \Delta x,
$$

hence $\Delta x^T \Delta s = 0$. Moreover, from (7) follows

$$
d_x^T d_s = e^T(d_x d_s) = \frac{1}{\mu}e^T(\Delta x \Delta s) = \frac{1}{\mu}\Delta x^T \Delta s = 0,
$$

thus d_x and d_s are orthogonal. We shall use this relation later in the paper. We conclude that in this section we have defined a class of search directions for the problem (SP). For this purpose we have used a function φ to transform the system (3) in an equivalent form. In the next section we shall consider a particular member of this class of search directions. Thus we shall develop a new polynomial algorithm for the self-dual problem (SP).

4. The Algorithm

In the remaining part of the paper we assume that $\varphi(x) = \sqrt{x}$. Using this function we present a new primal-dual interior-point algorithm for solving the problem (SP). Consequently, we obtain also a solution of (P) and (D). In this case applying Newton's method for the system (4) yields

$$M\Delta x = \Delta s,$$

$$\sqrt{\frac{s}{\mu x}}\Delta x + \sqrt{\frac{x}{\mu s}}\Delta s = 2\left(e - \sqrt{\frac{xs}{\mu}}\right). \tag{9}$$

For $\varphi(x) = \sqrt{x}$ we have

$$p_v = 2(e - v), \tag{10}$$

and we can define a proximity measure to the central path by

$$\sigma(x, \mu) = \frac{\|p_v\|}{2} = \|e - v\| = \left\| e - \sqrt{\frac{xs}{\mu}} \right\|,$$

where $\|\cdot\|$ denotes the Euclidean norm (l_2 norm). Let us introduce the notation

$$q_v = d_x - d_s$$

Now using that the vectors d_x and d_s are orthogonal we obtain

$$\|p_v\| = \|q_v\|,$$

therefore the proximity measure can be written in the form

$$\sigma(x, \mu) = \frac{\|q_v\|}{2}.$$

Moreover, we have

$$d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2} \quad \text{and} \quad d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{11}$$

The algorithm can be defined as follows.

Algorithm 4.1 *Let $\epsilon > 0$ be the accuracy parameter and $0 < \theta < 1$ the update parameter (default $\theta = \frac{1}{2\sqrt{n}}$).*

begin

  $x := e$; $\mu := 1$;

  while $n\mu > \epsilon$ **do begin**

    $\mu := (1 - \theta)\mu$;

Compute Δx *using (9);*
$x := x + \Delta x$;
end
end.

In the next section we shall prove that this algorithm solves the linear optimization problem in polynomial time.

5. Complexity analysis

In this section we are going to prove that Algorithm 4.1 solves the problem (SP) in polynomial time. In the first lemma we investigate under which conditions the feasibility of the full Newton step is assured. Let $x_+ = x + \Delta x$ and

$$s_+ = s(x_+) = M(x + \Delta x) + q = s + M\Delta x = s + \Delta s.$$

Using these notations we can state the lemma.

Lemma 5.1 *Let* $\sigma = \sigma(x, \mu) < 1$. *Then the full Newton step is strictly feasible, hence* $x_+ > 0$ *and* $s_+ > 0$.

Proof: For each $0 \leq \alpha \leq 1$ introduce the notation $x_+(\alpha) = x + \alpha\Delta x$ and $s_+(\alpha) = s + \alpha\Delta s$. Then we have

$$x_+(\alpha)s_+(\alpha) = xs + \alpha(s\Delta x + x\Delta s) + \alpha^2\Delta x\Delta s,$$

and from (6) and (7) we obtain

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) = v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s.$$

Furthermore, from (8) and (11) we get

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) = (1-\alpha)v^2 + \alpha(v^2 + vp_v) + \alpha^2\left(\frac{p_v^2}{4} - \frac{q_v^2}{4}\right).$$

Using (10) we find that

$$v^2 + vp_v = 2v - v^2 = e - (e-v)^2 = e - \frac{p_v^2}{4},$$

and this relation leads to

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) = (1-\alpha)v^2 + \alpha\left(e - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}\right). \tag{12}$$

Evidently, the inequality $x_+(\alpha)s_+(\alpha) > 0$ is satisfied if

$$\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty < 1,$$

where $\|\cdot\|_\infty$ denotes the Chebychev norm (l_∞ norm). We have

$$\left\|(1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}\right\|_\infty \le (1-\alpha)\frac{\|p_v^2\|_\infty}{4}+\alpha\frac{\|q_v^2\|_\infty}{4} \le$$

$$\le (1-\alpha)\frac{\|p_v\|^2}{4}+\alpha\frac{\|q_v\|^2}{4}=\sigma^2<1.$$

Hence, for each $0 \le \alpha \le 1$ we have $x_+(\alpha)s_+(\alpha) > 0$. Consequently, the sign of the continuous functions of α, $x_+(\alpha)$ and $s_+(\alpha)$ remains the same for every $0 \le \alpha \le 1$. Hence $x_+(0) = x > 0$ and $s_+(0) = s > 0$ yields $x_+(1) = x_+ > 0$ and $s_+(1) = s_+ > 0$. This completes the proof. ■

In the following lemma we formulate a condition which guarantees the quadratic convergence of the Newton process. We mention that in fact this requirement will be identical to that one used in Lemma 5.1, namely $\sigma(x,\mu) < 1$.

Lemma 5.2 *Let $\sigma = \sigma(x,\mu) < 1$. Then*

$$\sigma(x_+,\mu) \le \frac{\sigma^2}{1+\sqrt{1-\sigma^2}}.$$

Hence, the full Newton step is quadratically convergent.

Proof: We deduce from Lemma 5.1 that the full Newton step is strictly feasible, thus $x_+ > 0$ and $s_+ > 0$. Denote

$$v_+ = \sqrt{\frac{x_+s_+}{\mu}},$$

and observe that making the substitution $\alpha = 1$ in (12) that equation becomes

$$v_+^2 = e - \frac{q_v^2}{4}. \tag{13}$$

Thus

$$\min(v_+) = \sqrt{1-\frac{1}{4}\|q_v^2\|_\infty} \ge \sqrt{1-\frac{\|q_v\|^2}{4}} = \sqrt{1-\sigma^2}, \tag{14}$$

where for each vector ξ we denote $\min(\xi) = \min\{\xi_i \mid 1 \le i \le n\}$. Furthermore, (13) and (14) lead to

$$\sigma(x_+s_+,\mu) = \left\|\frac{e-v_+^2}{e+v_+}\right\| \le \frac{1}{1+\min(v_+)}\|e-v_+^2\| \le$$

$$\le \frac{1}{1+\sqrt{1-\sigma^2}}\left\|\frac{q_v^2}{4}\right\| \le \frac{1}{1+\sqrt{1-\sigma^2}}\frac{\|q_v\|^2}{4} = \frac{\sigma^2}{1+\sqrt{1-\sigma^2}}$$

Consequently, we have $\sigma(x_+s_+,\mu) < \sigma^2$, and this implies the lemma. ■

From the self-dual property of the problem (SP) follows that the duality gap is

$$2(q^Tx) = 2(x^Ts),$$

where x is a feasible solution of (SP), and $s = s(x)$ is the appropriate slack vector. For simplicity we also refer to x^Ts as the duality gap. In the following lemma we analyse the effect of the full Newton step on the duality gap.

Lemma 5.3 *Let $\sigma = \sigma(x,\mu)$ and introduce the vectors x_+ and s_+ such that $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$. Then we have*

$$(x_+)^T s_+ = \mu(n - \sigma^2).$$

Thus $(x_+)^T s_+ \le \mu n$.

Proof: Substituting $\alpha = 1$ in (12) results in

$$\frac{1}{\mu} x_+ s_+ = e - \frac{q_v^2}{4},$$

and using this equation we get

$$(x_+)^T s_+ = e^T(x_+s_+) = \mu(e^Te - \frac{e^Tq_v^2}{4}) = \mu(n - \frac{\|q_v\|^2}{4}) = \mu(n - \sigma^2)$$

This implies the lemma. ■

In the following lemma we investigate the effect on the proximity measure of a full Newton step followed by an update of the parameter μ. Assume that μ is reduced by the factor $(1-\theta)$ in each iteration.

Lemma 5.4 *Let $\sigma = \sigma(x,\mu) < 1$ and $\mu_+ = (1-\theta)\mu$, where $0 < \theta < 1$. We have*

$$\sigma(x_+,\mu_+) \le \frac{\theta\sqrt{n} + \sigma^2}{1 - \theta + \sqrt{(1-\theta)(1-\sigma^2)}}.$$

Furthermore, if $\sigma < \frac{1}{2}$ and $\theta = \frac{1}{2\sqrt{n}}$ then $\sigma(x_+,\mu_+) < \frac{1}{2}$.

Proof: From (13) and (14) we deduce

$$\sigma(x_+,\mu_+) = \left\| e - \sqrt{\frac{x_+s_+}{\mu_+}} \right\| = \frac{1}{\sqrt{1-\theta}} \left\| \sqrt{1-\theta}\,e - v_+ \right\| =$$

$$= \frac{1}{\sqrt{1-\theta}} \left\| \frac{(1-\theta)e - v_+^2}{\sqrt{1-\theta}\,e + v_+} \right\| \le \frac{1}{\sqrt{1-\theta}(\sqrt{1-\theta} + \min(v_+))} \left\| -\theta e + \frac{q_v^2}{4} \right\| \le$$

$$\le \frac{1}{1-\theta + \sqrt{(1-\theta)(1-\sigma^2)}} \left(\theta\sqrt{n} + \left\| \frac{q_v^2}{4} \right\| \right) \le \frac{\theta\sqrt{n} + \sigma^2}{1 - \theta + \sqrt{(1-\theta)(1-\sigma^2)}}.$$

Thus, the first part of the lemma is proved. Now observe that $n = m + k + 2 \geq 4$, hence for $\theta = \frac{1}{2\sqrt{n}}$ we get $1 - \theta \geq \frac{3}{4}$. Consequently, from $\sigma < \frac{1}{2}$ follows $\sigma(x_+, \mu_+) < \frac{1}{2}$. Thus the proof is complete. ∎

From Lemma 5.4 we conclude that the algorithm is well defined. Indeed, the requirements $x > 0$ and $\sigma(x, \mu) < \frac{1}{2}$ are maintained at each iteration. In the following lemma we discuss the question of the bound on the number of iterations.

Lemma 5.5 *Let x^k be the k-th iterate of Algorithm 4.1, and let $s^k = s(x^k)$ be the appropriate slack vector. Then $(x^k)^T s^k \leq \epsilon$ for*

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{n}{\epsilon} \right\rceil .$$

Proof: Using Lemma 5.3 we find that

$$(x^k)^T s^k \leq \mu^k n = (1-\theta)^k \mu^0 n = (1-\theta)^k n,$$

thus the inequality $(x^k)^T s^k \leq \epsilon$ is satisfied if

$$(1-\theta)^k n \leq \epsilon.$$

Now taking logarithms, we may write

$$k \log(1-\theta) + \log(n) \leq \log \epsilon,$$

and using the equation $-\log(1-\theta) \geq \theta$ we observe that the above inequality holds if

$$k\theta \geq \log(n) - \log \epsilon = \log \frac{n}{\epsilon}.$$

Thus the proof is complete. ∎

For $\theta = \frac{1}{2\sqrt{n}}$ we obtain the following theorem.

Theorem 5.6 *Let $\theta = \frac{1}{2\sqrt{n}}$. Then Algorithm 4.1 requires at most*

$$O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$$

iterations. ∎

6. Concluding remarks

In this paper we have developed a new class of search directions for the self-dual linear optimization problem. For this purpose we have introduced a function φ, and we have used Newton's method to define new search directions. For $\varphi(x) = \sqrt{x}$ these results can be used to introduce a new primal-dual polynomial algorithm for solving (SP). We have proved that the complexity of this algorithm is $O\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$.

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