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SEMANTICS FOR CONSTRAINED AND RATIONAL DEFAULT LOGICS

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ABSTRACT. The default nonmonotonic reasoning was formalised by a class of logical systems: default logics (classical, justified, constrained, rational), based on the same syntax which utilises nonmonotonic inference rules: defaults, but with different semantics for the defaults. In this paper we introduce a uniform semantic characterisation for the constrained and rational extensions of a default theory. This characterisation is an operational approach of the nonmonotonic reasoning that is viewed as a successive application of the applicable defaults. During the reasoning process can be observed the interaction between the defaults and the reasoning context. The graphical interpretation associated to the semantic characterisation of extensions illustrates the type of applicability: *cautious* (for constrained extensions) and *hazardous* (for rational extensions) of the defaults and some formal properties: semi-monotonicity, regularity, existence of extensions, commitment to assumptions of these variants of default logic.

1. INTRODUCTION

An important part of commonsense reasoning is default reasoning, which means drawing conclusions in the absence of complete information using default assumptions. This type of reasoning is nonmonotonic because the conclusions (formulas which are only plausible, not necessarily true) inferred can be later invalidated by adding new facts. Reiter [6] introduced nonmonotonic inference rules called *defaults*, which permit reasoning on the basis of "the lack of evidence to the contrary". The *classical default logic* was the first logical system that formalizes the default nonmonotonic reasoning.

A *default theory* is a pair (D, W), where W is a set of consistent formulas from first order logic and D is a set of default rules. W contains the facts (axioms) of the theory and D contains general rules that might have exceptions. A *default rule* has the form[1]:$d = \frac{\alpha:\beta}{\gamma}$, where $\alpha, \beta, \gamma$are formulas of first order logic,α is the *prerequisite* (*Prereq(d)*) of the default d, β is the *justification (Justif(d))* of the default d and γ is the *consequent (Conseq(d))* of the default d.

In this paper the followin g notations will be used: Justif(D)=$\bigcup_{d\in D} Justif(d)$, Prereq(D)= $\bigcup_{d\in D} \Pr ereq(d)$, Conseq(D)= $\bigcup_{d\in D} Conseq(d)$.

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[1]Due to the (semi) representability results for these versions of default logic, we use in this paper only defaults with at most one justification (unitary default theories).

A default $d = \frac{\alpha:\beta}{\gamma}$ can be applied and thus derive γ if α *is believed* and *it is consistent to assumed* β.

Using the classical inference rules and the defaults we can extend the initial set of facts with new formulas called *nonmonotonic theorems* obtaining *extensions.*

Definition 1.1[6]: Let (D,W) be a default theory. For any set of formulas S, let Γ(S) be the smallest set of formulas S' such that:

(1) W⊆S';
(2) Th(S')= S',whereTh(X) is the set of all the theorems obtained from the set X of formulas and using the classical inference rules;
(3) For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\alpha \in$S' and $\neg\beta \notin$S then $\gamma \in$S'. The application of the default rule means: if we can believe $\alpha$and $\neg\beta$ is not believed, then we can believe γ.

A set E of formulas is a *classical extension* of (D,W) if and only if Γ(E)=E.

A classical extension for a default theory is a maximal set of conclusions (beliefs) derived from the facts of W using classical derivation and the defaults as inference rules. A default theory may has zero, one or more classical extensions.

Due to the individual consistency checking of justifications and thus the loss of implicit assumptions when are constructed the classical extensions, this logical system does not satisfy some desirable formal properties: semi-monotonicity, regularity, existence of extensions, commitment to assumptions. In classical default logic these properties are satisfied only for normal default theories, that are the theories with all the defaults of the form: $\frac{\alpha:\beta}{\beta}$.

The next versions of default logic (justified default logic, constrained default logic, rational default logic) try to obtain for general default theories the above properties by modifying the meaning of the statement "*it is consistent to assumed* β**".**

Reiter has not provided a semantic for classical default logic, but he has observed that the application of defaults restricts the models of the initial set of facts W, and thus the class of models of an extension is a restricted class of models of W. This idea was formalised by Lukaszewicz [2] for normal default theories and then generalised by Etherington in [1]. Later on, as the new versions of default logics were defined, new approaches of the semantic of these logics appeared.

The *focused models semantics* was introduced for the classical default logic in [1] and then used in [7] for constrained default logic. This semantics is based on a preference between *focused models structures* induced by some set of default rules. An extension is characterized by a maximal models structure.

In the papers [7, 8] Schaub has developed a uniform semantical framework for all the variants of default logic. This approach is called *possible worlds semantics for default logics* and uses Kripke structures to characterise the two components of an extension: the set of beliefs and the underlying assumptions.

The aim of this paper is to provide a semantic characterization for constrained extensions and rational extensions of a default theory in the spirit of the approach of Lukaszewicz [3]. Thus we can obtain an uniform approach of the semantics for classical, justified, constrained, rational default logics, based on the idea that the reasoning process, viewed as a successive application of the applicable defaults, restricts the models of the initial set of facts. The advantage of this characterization is its graphical interpretation, which illustrates the semantics of applicability conditions for defaults and some formal properties.

The paper is organized as follows. Section 2 provides some notions and results about constrained and rational default logics. In Section 3 we propose an operational semantic characterization for constrained and rational extensions. Section 4 of the paper contains the graphical interpretation of the semantic characterization for constrained and rational extension. The last section is a comparative study of the formal properties of the variants (classical, justified, constrained, rational) of default logic from a semantical point of view.

2. Constrained and rational defaults logics

Schaub defined constrained default logic in [7] as an alternative approach to classical default logic. The nonmonotonic reasoning formalized by this logic is based on the observation that when we draw conclusions, we have to keep track of the assumption used in the inference process and then to verify that they do not contradict each other and do not contradict with the conclusions.

Definition 2.1[7]: Let (D,W) be a default theory. For any set of formulas T, let Ψ(T) be the pair of the smallest sets of formulas (S',T') such that:

(1) W⊆S'⊆T';
(2) S'=Th(S') and T'=Th(T');
(3) For $\frac{\alpha:\beta}{\gamma} \in D$, if ∀0S' and $T \cup \{\beta\} \cup \{\gamma\}$ is consistent, then γ ∈S' and$\beta \wedge \gamma$ ∈T'.

A pair (E,C) of sets of formulas is a *constrained extension* of (D,W) if and only if Ψ(E,C)=(E,C).

The set E is the *actual extension* of the default theory and C is the *set of constraints* (a consistent *context* for E), which keeps track of the justifications assumed to be true in the construction of E.

The *set of the generating defaults* for the constrained extension (E,C) is defined as follows:$GD_{\Delta}^{(E,C)} = \left\{ \frac{\alpha:\beta}{\gamma} | \alpha \in E \; and \; C \cup \{\beta, \gamma\} \not\mapsto fals \right\}$.

This logical system satisfies the properties: semi-monotonicity, the existence of a constrained extension is guaranted, strong-regularity and commitment to assumption.

Rational default logic was proposed in [5] and is based on the idea that we cannot use in the construction of an extension defaults whose all justifications together are inconsistent with the extension.

Definitions 2.2[5]: Let (D,W) be a default theory, let X be a subset of the set D of defaults and let S be a set of formulas.

(1) We define $X_S = \left\{ \frac{\alpha}{\gamma} | \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in X, \; S \cup \{\neg\beta_i\} \; inconsistent, \; 1 \leqslant i \leqslant n \right\}$ and Mon(X)=$\left\{ \frac{\Pr ereq(d)}{Conseq(d)} | d \in X \right\}$.
(2) A set X of defaults is *active* with respect to W and S if it satisfies the conditions:
   (i) Justif(X)=∅ or Justif(X)∪ S is consistent
   (ii) Prereq(X)⊆ Th^{X_S}(W), where Th^{X_S}(W) is the deductive closure of W using classical inference rules and the monotonic rules fromX_S.

(3) We denote by A(D,W,S) the set of all subsets of the defaults in D which are active with respect to W and S. $\emptyset \subseteq$ A(D,W,S). MA(D,W,S) is defined as the set of all maximal elements in A(D,W,S).

The set E of formulas is a *rational extension* for the theory (D,W) if E=Th^{X_E}(W), where X$\in$ MA(D,W,E), and X is the *set of generating defaults*.

The next definition is proposed by Schaub [8] and it is equivalent with the original definition of rational default logic.

Definition 2.3[8]: Let (D,W) be a default theory. For any set of formulas T, let Ψ(T) be the pair of the smallest sets of formulas (S',T') such that:

(1) W$\subseteq$S'$\subseteq$T';
(2) S'=Th(S') and T'=Th(T');
(3) For any $\frac{\alpha:\beta}{\gamma} \in D$, if $\forall 0$S' and $T \cup \{\beta\}$ is consistent, then $\gamma \in$S' and$\beta \wedge \gamma \in$T'.

A pair (E,C) of sets of formulas is a *rational extension* of (D,W) if and only if Ψ(E,C)=(E,C). The set E is the actual extension of the default theory and C is the reasoning context.

The *set of the generating defaults* for the rational extension (E,C) is defined as follows: $GD_{\Delta}^{(E,C)} = \left\{ \frac{\alpha:\beta}{\gamma} | \alpha \in E \; si \; C \cup \{\beta\} \not\mapsto fals \right\}$.

Rational default logic is a generalisation of constrained default logic that means: each rational extension of a default theory is also a constrained extension of the same theory. It can be easy observed that the sets of generating defaults for constrained extensions are active sets, not necessarily maximal active sets. Constrained default logic and rational default logic coincide on the class of semi-normal default theories. This logical system is strongly regular, does not guarantee the existence of extensions, is not semi-monotonic and does not commit to assumptions.

The following theorems provide characterisations for constrained, respective rational extensions of a default theory, using the set of generating defaults.

Theorem 2.1[8]: Let (D,W) be a default theory and let E and C be sets of formulas. Then (E,C) is a constrained extension of (D,W) if and only if E=Th(W$\cup$Conseq(D')) and C=Th(W$\cup$Justif(D')$\cup$Conseq(D')) for a maximal set D'$\subseteq$D such that D' is grounded in W and W$\cup$Justif(D')$\cup$Conseq(D') is consistent.

Theorem 2.2[4]: Let (D,W) be a default theory and let E and C be sets of formulas. Then (E,C) is a rational extension of (D,W) if and only if E=Th(W $\cup$ Conseq(D')) and C=Th(W $\cup$ Justif(D') $\cup$ Conseq(D')) for a maximal D' $\subseteq$ D such that D' is grounded in W and are satisfied the following conditions:

(i) W$\cup$Conseq(D')$\cup$Justif(D') is consistent
(ii) $\forall d \in$ D\D' we have: W$\cup$Conseq(D')$\cup$\{$\neg$Precond(d)\} is consistent or W$\cup$Conseq(D')$\cup$Justif(D'$\cup$\{d\}) is inconsistent.

The condition (ii) from the above theorem states that the set of generating defaults is maximal active with respect to W and E.

The two variants of the default logic presented in this section have as a common feature the fact that the inference process formalised by them is guided by a reasoning consistent context, which contains the beliefs and the underlying assumptions, used for deriving new nonmonotonic theorems.

3. Semantic characterization for constrained extensions and rational extensions of a default theory

In this section we will provide semantic characterization for constrained and rational extensions of a default theory. This operational approach is inspired from [3], the construction of an extension is viewed as a successive application of the applicable defaults.

In the following some notions from the semantic of first order logic will be used.

Definition 3.1:

(1) For each class of frames $\Lambda$and a formula A we denote by Λ(A)={M|M$\in$ $\Lambda$and $|=_M$A} the models of A, that means the set of all the frames from Λ in which the formula A is true.
(2) The frame M is a *model* for the set S of formulas, if and only if are satisfied: $|=_M$A, $\forall A \in S$. We will use the notation $|=_M$S.
(3) A *class* Λ of frames is *elementary* if and only if Λ is the class of all the models of the set S of formulas.

The particularity of these two variants of default logic that are used implicit assumptions to derive new explicit conclusions suggests that there is an explicit content (the set of beliefs) and an implicit content (assumptions) of the knowledge base. These two aspects must be correlated in the semantic characterisation of an extension.

Constrained and rational extensions are defined using a pair (E=actual extension, C=reasoning context). Thus it is naturally to have a pair $<\Lambda_1,\Lambda_2>$ which characterises semantic these types of extensions as follows: Λ_1 is the class of all the models of the set E of beliefs, and Λ_2 is the class of all the models of the context C. We have $\Lambda_2 \subseteq \Lambda_1$ since E$\subseteq$C.

Definition 3.2: Let Λ_1 and Λ_2 be two classes of frames. The pair $<\Lambda_1,\Lambda_2>$ is called a bi-*structure* if and only if Λ_1 and $\Lambda_2$are elementary classes and $\Lambda_2 \subseteq \Lambda_1$.

A bi-structure $<Lambda_1,\Lambda_2>$ characterises the stage of the reasoning process as follows: $\Lambda_1$represents the set of all the models of a set of beliefs and Λ_2 represents the set of all the models of a set of formulas, which represent the reasoning context.

reasoning context = set of beliefs (non-monotonic theorems) +
implicit assumptions (justifications of the used defaults)

Definition 3.3: Let $<\Lambda_1,\Lambda_2>$ be a pair of frames-frames and $d = \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma}$ a default.

(1) The default d is *res-applicable with respect to* $<\Lambda_1,\Lambda_2>$ if and only if:
(i) $|=_M \alpha$, $\forall M \in \Lambda_1$ and (ii) $\exists M \in \Lambda_2$ *a.i.* $|=_M \beta \wedge \gamma$
(2) The default d is *rat-applicable with respect to* $<\Lambda_1,\Lambda_2>$ if and only if:
(i) $|=_M \alpha$, $\forall M \in \Lambda_1$ and (ii) $\exists M \in \Lambda_2$ *a.i.* $|=_M \beta$

The conditions of res-applicability and rat-applicability are the semantic counterpart of the applicability conditions from definitions 2.1 and 2.3. We can interpret these semantic conditions of applicability as follows:

- res-applicable with respect to $<\Lambda_1, \Lambda_2>$ if and only if the prerequisite is believed and the justification together with the consequent are consistent with the reasoning context.
- rat-applicable with respect to $<\Lambda_1, \Lambda_2>$ if and only if the prerequisite is believed and the justification is consistent with the reasoning context.

Definition 3.4: To a closed default $d = \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma}$ we assign a mapping d^{res} from the set of the bi-structures into the set of bi-structures as follows:

$$d^R(<\Lambda_1,\Lambda_2>) = \begin{cases} <\Lambda_1(\gamma),\Lambda_2(\beta\wedge\gamma)> & \text{if } d \text{ is res-applicable wrt } <\Lambda_1,\Lambda_2> \\ <\Lambda_1,\Lambda_2> & \text{otherwise} \end{cases}$$

This mapping models a *cautious application* of the defaults which means that the commitment to assumptions for each applied default is guaranteed, and then inconsistencies after the application of an applicable default cannot be obtained.

Definition 3.5: To a closed default $d = \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma}$ we assign a mapping d^{rat} from the set of frames-frames into the set of frames-frames as follows:

$$d^{rat}(<\Lambda_1,\Lambda_2>) = \begin{cases} <\Lambda_1(\gamma),\Lambda_2(\beta\wedge\gamma) & \text{if } <\Lambda_1,\Lambda_2> \text{ is a bi-structure and} \\ & d \text{ is rat-aplicable wrt } <\Lambda_1,\Lambda_2> \\ <\Lambda_1,\Lambda_2> & \text{if } <\Lambda_1,\Lambda_2> \text{ is a bi-structure and} \\ & d \text{ is not rat-aplicable wrt } <\Lambda_1,\Lambda_2> \\ <\emptyset,\emptyset> & \text{if } <\Lambda_1,\Lambda_2> \text{ is not a bi-structure} \end{cases}$$

The above definition models a step in the reasoning process, where the commitment to assumptions is not guaranteed. We say that we have a *hazardous application* of the defaults that means: the application of an applicable default can cause inconsistencies in the set of beliefs or in the reasoning context.

Definition 3.6: Let $<\Lambda_1,\Lambda_2>$ be a bi-structure and D a set of closed defaults.

(1) $<\Lambda_1,\Lambda_2>$ *is res-stable wrt D* if and only if $d^{res}(<\Lambda_1,\Lambda_2>)=<\Lambda_1,\Lambda_2>$, $\forall d\in D$.

(2) $<\Lambda_1,\Lambda_2>$ is *rat-stable wrt D* if and only if $d^{rat}(<\Lambda_1,\Lambda_2>)=<\Lambda_1,\Lambda_2>$, $\forall d\in D$.

A stable bi-structure characterises the end of the reasoning process in which all the applicable defaults were used.

Definition 3.7: Let $<\Lambda_1,\Lambda_2>$ be a pair of frame-frame and $<d_i>$ a sequence of closed defaults.

(1) We denote by $<d_i>^{res}(<\Lambda_1,\Lambda_2>)$ the bi-structure obtained as follows:

$$<d_i>^{rat}(<\Lambda_1,\Lambda_2>) = \begin{cases} <\Lambda_1,\Lambda_2> & \text{if } <d_i>=\emptyset \\ <\cap\Lambda_1^i,\cap\Lambda_2^i> & \text{else} \end{cases}$$

where $<\Lambda_1^0,\Lambda_2^0>=<\Lambda_1,\Lambda_2>$, and $<\Lambda_1^{i+1},\Lambda_2^{i+1}>=d_i^{res}(<\Lambda_1^i,\Lambda_2^i>)$ for i=1,2,...

(2) We denote $<d_i>^{rat}(<\Lambda_1,\Lambda_2>)$ the pair frames-frames obtained as follows:

$$<d_i>^{res}(<\Lambda_1,\Lambda_2>) = \begin{cases} <\Lambda_1,\Lambda_2> & \text{if } <d_i>=\emptyset \\ <\cup\Lambda_1^i,\cup\Lambda_2^i> & \text{else} \end{cases}$$

where $<\Lambda_1^0,\Lambda_2^0>=<\Lambda_1,\Lambda_2>$, and $<\Lambda_1^{i+1},\Lambda_2^{i+1}>=d_i^{rat}(<\Lambda_1^i,\Lambda_2^i>)$ for i=1,2,...

These definitions model a reasoning process that consists in a successive application of the elements from a sequence of defaults.

Definition 3.8: Let $< \Lambda_1,\Lambda_2 >$ be a bi-structure, let Z be an elementary class of frames and $< d_i >$ a sequence of closed defaults. The pair $< \Lambda_1,\Lambda_2 >$ is $< d_i >$-x-*accessible from Z* if and only if $< \Lambda_1,\Lambda_2 >=< d_i >^x(<$ Z,Z$>)$. $< \Lambda_1,\Lambda_2 >$ is x-*accessible* from Z *wrt D* if and only if there exists a sequence $< d_i >$ of defaults in D such that $< \Lambda_1,\Lambda_2 >$ is $< d_i >$-x-accessible from Z. x can be *res* or *rat*, and thus the notions of *res-accessibility* and *rat-accessibility* are defined.

Using the notions presented before, the following theorems provide semantic characterisations for constrained extensions, respective rational extensions.

Theorem 3.1 (correctness and completeness): Let (D,W) be a closed default theory and let Z be the class of all models of W. A class of frames Λ_1 is the class of all models of actual extension E and $\Lambda_2$is the class of all models of the reasoning context C (where (E,C) is a constrained extension of (D,W)) if and only if there exists a bi-structure $< \Lambda_1,\Lambda_2 >$ which satisfies:

(i) $< \Lambda_1,\Lambda_2 >$ is res-stable wrt D and (ii) $< \Lambda_1,\Lambda_2 >$ is res-accessible from Z wrt D.

Proof:

(**correctness**) Assume that (E,C) is a constrained extension for (D,W), then according to theorem 2.1 we have that:

E=Th(W∪Conseq(D')), C=Th(W∪Conseq(D')∪Justif(D')), where D' is grounded in W and the set W∪Conseq(D') ∪Justif(D') is consistent.

Let Λ_1={M| $\ |=_M W \cup Conseq(D')\}$ be the set of all models of the actual extension E and let Λ_2={M| $\ |=_M W \cup Conseq(D') \cup Justif(D')\}$ be the set of all models of the reasoning context C. We have then that $< \Lambda_1,\Lambda_2 >$ is a bi-structure and we have to verify that $< \Lambda_1,\Lambda_2 >$ satisfies conditions (i) and (ii) from the theorem:

- For (i) we have to prove that $\forall d \in D : d^{res}(< \Lambda_1,\Lambda_2 >)=< \Lambda_1,\Lambda_2 >$

There are two cases:

1. For $d = \frac{\alpha:\beta}{\gamma} \in D'$:

$\alpha \in$E, hence $|=_M \alpha$, $\forall M \in \Lambda_1$ **and**

$\gamma \in$E, C$\cup\{\beta \wedge \gamma\}$ is consistent, hence $\exists M \in \Lambda_2$ *such that* $\ |=_M \beta \wedge \gamma$

We have $\Lambda_1(\gamma) = \Lambda_1$ since $\gamma \in$E, and $\Lambda_2(\beta \wedge \gamma)$¿ $=\Lambda_2$ since$\beta \wedge \gamma \in$C.

The default d is res-applicable wrt $< \Lambda_1,\Lambda_2 >$ according to definition3.3 and $d^{res}(< \Lambda_1,\Lambda_2 >)= < \Lambda_1(\gamma),\Lambda_2(\beta \wedge \gamma) >= < \Lambda_1,\Lambda_2 >$

2. For $d = \frac{\alpha:\beta}{\gamma} \in D\backslash D'$:

$\alpha \notin$E, hence $\exists M \in \Lambda_1$ *such that* $\ |\not=_M \alpha$ **or**

C$\cup\{\beta \wedge \gamma\}$ is inconsistent, hence $\nexists M \in \Lambda_2$ *such that* $\ |=_M \beta \wedge \gamma$

According to the definition 3.3 the default d is not res-applicable wrt $< \Lambda_1,\Lambda_2 >$ and $d^{res}(< \Lambda_1,\Lambda_2 >)=< \Lambda_1,\Lambda_2 >$

Thus we have proved the res-*stability* of the *bi-structure* $< \Lambda_1,\Lambda_2 >$ wrt D.

- For (ii) we have to prove that there exists a sequence of defaults $< d_i >$ such that $< \Lambda_1,\Lambda_2 >=< d_i >^{res}(<Z,Z>)$, where Z={M| $\ |=_M$ W}.

The set D' is grounded in W, therefore exists an enumeration $\langle \delta_i \rangle_{i \in I}$ of its elements such that:

(1) $W \cup Conseq(\{\delta_0, \delta_1, \ldots, \delta_{i-1}\}) \mapsto \Pr econd(\delta_i), \forall i \in I = \{0, 1, 2, n\}$

We consider that this enumeration represents the sequence $< d_i >$, which provides the application order of the defaults for generating the constrained extension (E,C).

a) if $< d_i >= \emptyset$ then $< \Lambda_1, \Lambda_2 >=<Z,Z>$, the set of generating defaults D'=$\emptyset$ and the constrained extension (E,C), E=C=Th(W).
b) if $< d_i > \neq \emptyset$ we show by induction that:

(2) $< d_0, ..., d_k >^{res} (< Z, Z >) =< \Lambda_1^{k+1}, \Lambda_2^{k+1} >, k = 0, \ldots, n,$ where

Λ_1^{k+1} ={M| $\ |=_M W \cup Conseq(\{d_0, ..., d_k\})\}$ and

Λ_2^{k+1} ={M| $\ |=_M W \cup Conseq(\{d_0, ..., d_k\}) \cup Justif(\{d_0, ..., d_k\})\}$

Base: k=0

From (1) with i=0 we have: W$\mapsto$ $\Pr econd(d_0)$ which implies W$|=$ $\Pr econd(d_0)$, therefore $\forall M \in Z \quad |=_M \Pr econd(d_0)$. The set W$\cup$Conseq($d_0$) $\cup$Justif(d_0) is consistent because is a subset of the consistent set W$\cup$Conseq(D')$\cup$Justif(D')).

Hence $\exists M \in Z$ *such that* $|=_M Justif(d_0) \wedge Conseq(d_0)$.

Thus are satisfied the res-applicability conditions for the default d_0 wrt <Z,Z>.

d_0^{res}(<Z,Z>)= $< \Lambda_1^1$=Z(Conseq(d_0),Λ_2^1=Z(Justif(d_0)$\wedge$Conseq(d_0)) $>$.

Step: Let us assume that the relation (2) is true for k and we will prove that it is true for k+1.

From (1) with i=k we have: W$\cup$Conseq($\{d_0, \ldots, d_k\}$)$\mapsto$ $\Pr econd(d_{k+1})$ which implies W$\cup$Conseq($\{d_0, \ldots, d_k\}$)$| = \Pr econd(d_{k+1})$, and then $\forall M \in \Lambda_1^{k+1} \quad |=_M$ $\Pr econd(d_{k+1})$.

W$\cup$Conseq($\{d_0, ..., d_k, d_{k+1}\}$)$\cup$Justif($\{d_0, \ldots, d_k, d_{k+1}\}$) is a consistent set because is a subset of the consistent set W$\cup$Conseq(D')$\cup$Justif(D').

Hence $\exists M \in \Lambda_2^{k+1}$ *such that* $|=_M Justif(d_{k+1}) \wedge Conseq(d_{k+1})$

Thus are satisfied the res-applicability conditions for the default d_{k+1} wrt $<$ Λ_1^{k+1},Λ_2^{k+1} $>$.

$< d_0, ..., d_{k+1} >^{res}$(<Z,Z>) = d_{k+1}^{res}(¡Λ_1^{k+1},Λ_2^{k+1} $>$)= $< \Lambda_1^{k+1}$ (Conseq(d_{k+1})), Λ_2^{k+1} (Conseq(d_{k+1})$\wedge$ Justif d_{k+1})) $>$ = $< \Lambda_1^{k+2}$, Λ_2^{k+2} $>$ and thus the relation (2) is satisfied for k+1.

For i=n we have: $< d_0, \ldots, d_n >^{res}$ (<Z,Z>) = $< \Lambda_1^{n+1}$,Λ_2^{n+1} $>$ = <{M|$| =_M W \cup Conseq(\{d_0, \ldots, d_n\})\}$, {M|$| =_M W \cup Conseq(\{d_0, \ldots, d_n\}) \cup Justif(\{d_0, \ldots, d_n\})\}$¿ = $< \Lambda_1$, $\Lambda_2 >$ since D'=$\{d_0, ..., d_n\}$.

Thus was proved the *res-accessibility* of the *bi-structure* $< \Lambda_1$, $\Lambda_2 >$ from Z wrt D.

(**completeness**) We assume that $< \Lambda_1$,$\Lambda_2 >$ is a bi-structure which satisfies the conditions (i) and (ii). Λ_1 is the set of all the models of the set E of formulas and $\Lambda_2$is the set of all the models of the set C of formulas. The relation $\Lambda_1 \supseteq \Lambda_2$ implies E$\subseteq$C.

We have to prove that (E,C) is a constrained extension for the default theory (D,W).

According to the condition (ii) there exists a sequence of defaults $< d_i >= d_0, ..., d_n$in D such that $< \ d_0, ..., d_n \ >^{res}$(<Z,Z>)= $<$ Λ_1,Λ_2 $>$**,** where Z={M| $\ | =_M$W}

If $< d_i >= \emptyset$ **then** $< \Lambda_1$,$\Lambda_2 >$ = <Z,Z> .

Since <Z,Z> is a bi-structure res-stable wrt D we have that (E=Th(W),C=Th(W)) is a constrained extension for (D,W). The set of the generating defaults is $\emptyset$.

If $< d_i >\neq \emptyset$ **then**:

Following step by step the application of the defaults in the sequence we observe that:

(a0) $| =_M$Precond(d_0), $\forall M \in Z = \Lambda_1^0$ and thus W$\mapsto$ $\Pr econd(d_0)$

(b0) d_0^{res}(<Z,Z>)=$< \Lambda_1^1$ = Z(Conseq(d_0), Λ_2^1 = Z(Justif(d_0)$\wedge$Conseq(d_0)) $>$ = <{M| $\ | =_M W \cup Conseq(d_0)\}$, {M| $\ | =_M W \cup Conseq(d_0) \cup Justif(d_0)\}$¿.

$< \Lambda_1^1$,$\Lambda_2^1 >$ is a bi-structure which implies that W$\cup$Conseq(d_0)$\cup$Justif(d_0) is a consistent set.

Using the notations:

Λ_1^{k+1} ={M| $\ | =_M W \cup Conseq(\{d_0, ..., d_k\})\}$ and

Λ_2^{k+1} ={M| $\ | =_M W \cup Conseq(\{d_0, ..., d_k\}) \cup Justif(\{d_0, ..., d_k\})\}$

we can easily prove by induction that for k=1,2,. . . , n are satisfied the relations:

(ak) $|\ =_M$Precond(d_k), $\forall M \in \Lambda_1^k$, hence W$\cup$Conseq($\{d_0, ..., d_k\}$)$\mapsto$ $\Pr econd(d_{k+1})$

(bk) $< d_0, ..., d_k >^{res}$(<Z,Z>)=$< \Lambda_1^{k+1}$, $\Lambda_2^{k+1} >$ is a bi-structure.

From (a0) + (a1) +. . . +(an) we have that the set D'=$\{d_0, ..., d_n\}$ is grounded in W.

For k=n we have: $< d_0, ..., d_n >^{res}$(<Z,Z>)= $< \Lambda_1$,$\Lambda_2 >$ = <{M| $\ | =_M$ W $\cup$Conseq(D')}, {M| $\ | =_M$ W $\cup$Conseq(D') $\cup$Justif(D')}¿

$< \Lambda_1$,$\Lambda_2 >$ is a bi-structure and thus W$\cup$Conseq(D')$\cup$Justif(D') is a consistent set.

The res-stability condition for $< \Lambda_1$,$\Lambda_2 >$ means that $d^{res}(< \Lambda_1$,$\Lambda_2 >$)=$<$ Λ_1,$\Lambda_2 >$, for every d in D :

If d$\in$D' then d is res-applicable wrt $< \Lambda_1$,$\Lambda_2 >$ but it was applied already.

If d$\in$D$\setminus$D' then we have two cases (i) or (ii):

(i) *d* is res-applicable wrt $< \Lambda_1$,$\Lambda_2 >$ but applying it we can neither obtain new nonmonotonic theorems nor modify the context. We add to D' all the defaults d with this property: D'=D'$\cup$\{d\} and D' remains grounded in W.

(ii) *d* is not res-applicable wrt $< \Lambda_1$,$\Lambda_2 >$ due to the following:

$\exists M \in \Lambda_1 \ a.i. \ | \not=_M \Pr econd(d)$,thus D'$\cup$\{d\} is not grounded in W **or**

$\nexists M \in \Lambda_2 \ a.i \ | =_M Justif(d) \wedge Conseq(d)$, thus

W$\cup$Conseq(D'$\cup$\{d\})$\cup$Justif(D'$\cup$\{d\}) is an inconsistent set.

All the defaults in D$\setminus$D' are not res-applicable wrt $< \Lambda_1$,$\Lambda_2 >$ and thus is guaranteed the maximality of D' such that D' is grounded in W and W$\cup$Conseq(D')$\cup$Justif(D') is consistent.

The maximal sets E, C of formulas with the property that Λ_1,Λ_2 are the sets of all their models respectively, have the form: E=Th(W$\cup$Conseq(D'), C=Th(W$\cup$Conseq(D')$\cup$Justif(D')) and thus (E,C) is a constrained extension according to the theorem 2.1.

Theorem 3.2 (correctness and completeness): Let (D,W) be a closed default theory and Z be the class of all models of W. A class of frames Λ_1 is the class of all the models of the actual extension E and $\Lambda_2$is the class of all the models of the reasoning context C (where (E,C) is a rational extension of the theory (D,W)) if and only if there exists a bi-structure $< \Lambda_1,\Lambda_2 >$ which satisfies the following conditions:

(i) $< \Lambda_1,\Lambda_2 >$ is rat-stable wrt D and (ii) $< \Lambda_1,\Lambda_2 >$ is rat-accessible from Z wrt D.

Proof: The proof of this theorem is similar to the above proof. Theorem 2.2 can be used for the characterization of rational extensions.

4. Graphical interpretation of the semantic characterization of constrained and rational extensions

To each default theory (D,W) we can associate a transition network. The nodes of this network contain pairs frames-frames and the arcs are labelled with defaults from the set D. If a node contains a bi-structure is called *viable*, otherwise is called *contradictory*. A *leaf node* is a node whose outbound arcs loop back.

The network is specified as follows:

(1) The set of nodes is the smallest set which satisfies the conditions:
   - $<$ Z, Z ¿ is the root node, where Z={M| $\;|=_M$W}.
   - if **n** is a viable node and d∈D, then d^{res}(n) (respective d^{rat}(n)) is a node of the network.

(2) From each viable node **n** and for each d∈D, there is an arc label by d which leads to the node d^{res}(n) (respective d^{rat}(n))

We can give a graphical interpretation for the theorems 3.1 and 3.2.

Let (D,W) be a default theory and the associated transition network built using d^{res}(respective d^{rat}). Every viable node characterises a *constrained extension* (respective *rational extension)* for the default theory (D,W) as follows:

$< \Lambda_1,\Lambda_2 >$ is a stable bi-structure contained in a leaf node, where:

$< \Lambda_1,\Lambda_2 >=< d_i >^{res}(<$ Z,Z$>)$ (respective $< \Lambda_1,\Lambda_2 >=< d_i >^{rat}(<$ Z,Z$>)$),

$< d_i >=d_1,\ldots,d_k$, and Z={M| $\;|=_M$W}.

if and only if

$\Lambda_1$is the class of all models of the set E=Th($W \cup \bigcup_{i=1}^{k} Conseq(d_i)$) and Λ_2 is the class of all the models of the set of formulas C=Th($W \cup \bigcup_{i=1}^{k} Conseq(d_i) \cup \bigcup_{i=1}^{k} Justif(d_i)$), that means:

$$\Lambda_1 = \{M|| =_M W \cup \bigcup_{i=1}^{k} Conseq(d_i)\},$$

$$\Lambda_2 = \{M|| =_M W \cup \bigcup_{i=1}^{k} Conseq(d_i) \cup \bigcup_{i=1}^{k} Justif(d_i)\}$$

and $\{d_1, \ldots, d_k\}$ is the set of generating defaults for the constrained extension (E,C) (respectively for the rational extension (E,C)).

Example 4.1: The default theory $(\left\{d1 = \frac{:B}{C}, d2 = \frac{:\neg B}{D}, d3 = \frac{:\neg C \wedge \neg D}{E}\right\}, \emptyset)$ has three constrained extensions corresponding to the leaf nodes of the transition network from the fig1.

(E1=Th({C}),C1=Th({C∧B})) with {d1} as the set of generating defaults.

(E2=Th({D}),C2=Th({D∧¬B})) with {d2} as the set of generating defaults.

(E3=Th({E}),C3=Th({E∧¬C∧¬D})) with {d3} as the set of generating defaults.

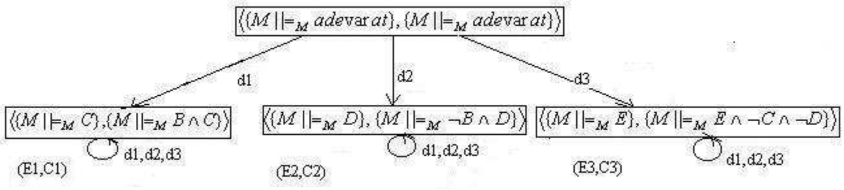

FIGURE 1.

Fig2 contains the transition network that characterizes the rational extensions. The same default theory has only two rational extensions: (E1,C1) and (E2,C2).

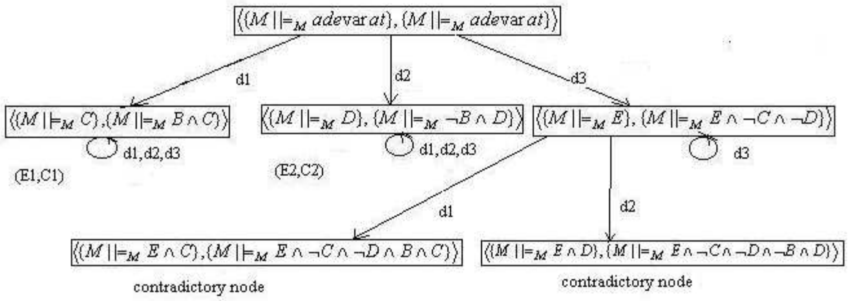

FIGURE 2.

According to the original definition of rational default logic we have that:

- {d1}∈ MA(D,W,E1), where E1=Th(W∪Conseq({d1})).
- {d2}∈ MA(D,W,E2), where E2=Th(W∪Conseq({d2})).
- {d1,d3},{d2,d3}∈ MA(D,W,S), where S=Th(W∪Conseq({d3})). Therefore the set {d3} is not maximal active with respect to W and S; hence {d3} cannot be a set of generating defaults for a rational extension.

The existence of the contradictory node in fig2 illustrates the semantical failure of semi-monotonicity, and thus the failure of commitment to assumption for rational default logic.

5. Conclusions

The semantic characterization of constrained and rational extensions proposed in this paper together with the semantic characterization of classical and justified extensions from [3] can be viewed as a uniform approach of semantics which permits a comparative study of the formal properties of these variants of default logic. We can observe similarities between classical and rational default logics, respective justified and constrained default logic as follows:

justified - constrained default logics:

- cautious application of the defaults
- a transition network which models the reasoning process in justified or constrained default logic does not have contradictory nodes
- these variants of default logic satisfy the semi-monotonicity property which permits to successively apply one default after another with no risk of destroying any previous partial extension
- a default theory has always justified and constrained extensions.

classical - rational default logics

- hazardous application of the defaults
- there is the possibility to obtain contradictory nodes, which means: a default that satisfies the applicability condition, after its application can cause inconsistencies in the set of beliefs or in the reasoning context
- the failure of the semi-monotonicity property, and thus a classical and a rational extension can not be generate iteratively
- the existence of classical and rational extensions is not guaranteed: there are transition networks having only contradictory nodes as final nodes
- does not commit to assumptions

Constrained default logic satisfies the property of commitment to assumptions due to the fact that the res-applicability condition is cautious and the reasoning context must be consistent.

Rational and constrained default logics are strong-regular because the applicability conditions for defaults require the reasoning context to be consistent.

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