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NUMERICAL SOLUTION OF THE DELAY DIFFERENTIAL EQUATIONS BY NONPOLYNOMIAL SPLINE FUNCTIONS

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Abstract. In this paper the nonpolynomial spline function to approximate the solution of the delay differential equations is constructed. The stability and the convergence of the nonpolynomial spline algorithms are also investigated.

Key Words: nonpolynomial splines, delay equations, collocation.

1. Introduction

The delay differential equations provide realistic models for many phenomena arising in applied mathematics. As known, delay differential equations can be used for the modeling of population dynamics, the spread of infections diseases, two-body problems of electrodynamics, etc.

Delay differential equations, or generally functional differential equations have been extensively studied in the past decades, especially as models to describe many physical and biological systems. Although, there exist several methods to solve numerically the delay differential equations, most of them cannot handle some difficulties properly. At the same time spline functions have steadily advanced to the front position as a very useful tool in general for the approximate the solutions of nonlinear differential, integral and partial differential equations, and particularly of the modified argument differential equations.

Comprehensive bibliography of published papers in this field (see [15]) and especially the reference therein.

The aim of this paper is to propose an alternative approximate method for the numerical solution of the delay differential equation problem and to introduce a new approach to the stability analysis of the nonpolynomial spline approximate

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solution. We will only consider in this paper the following scalar delay differential equation problem:

$$
\begin{aligned}
y'(t) &= f(t, y(t), y(g(t))), t \in [0, T] \\
y(t) &= \varphi(t), y'(t) = \varphi'(t), t \in [\alpha, 0], \alpha < 0
\end{aligned}
\tag{1}
$$

Several other methods for such delay differential equation problems have been proposed and their convergence has been investigated. For example, Oberle and Pesch [16] investigated the convergence of a numerical method for a constant delay $g(t) = t - h$, Bellen and Zennaro [4] investigated the convergence of a numerical method for time dependent delay, Tavernini [18], Arndt [2], Feldstein and Neves [8], Karoui and Vaillancourt [12, 13] and Baker and Paul [3] investigated the convergence of a numerical methods for state dependent delay differential equations. Jackiewicz [9, 10, 11] investigated the convergence of numerical method for time dependent delay of neutral delay differential equations. Recently, Enright and Hayashi [7] gave a very deep investigation of the convergence analysis of the solution of retarded and neutral delay differential equations by continuous numerical method.

We assume the existence, uniqueness and stability of the solutions to the mathematical problem under consideration. For example, sufficient conditions for the existence and uniqness of solution to the delay differential equation problem (1) are:

-f is continuous with respect to $t, y(t), y(g(t))$,

-$y(t)$ is continuous,

-f satisfies a Lipschitz condition in the last two argument,

-φ is continuous and

-f is bounded (see Driver[6]).

We suppose here that $f \in C^r([0, T] \times R^2, R)$ for a given natural number $r \in N$ and we shall introduce the nonpolynomial spline function space of degree $m \in N$ denoted by $S_m(\Delta)$, in which we shall find an approximate solution for the problem (1). It will be shown that our approximating method is a one-step method and the order of the method is $O(h^{\beta+r+m})$ in $y^{(q)}, (q = 0, 1, ..., r + 1), 0 < \beta \leq 1$.

Here m is an arbitrary positive integer, which in fact indicates the number of iteration processes in the method that describes the spline function.

Assume that f satisfies the following Lipschitz condition:

$$
\left| f^{(q)}(t, u_1, v_1) - f^{(q)}(t, u_2, v_2) \right| \leq L \left[|u_1 - u_2| + |v_1 - v_2| \right]
\tag{2}
$$

where $(t, u_1, v_1), (t, u_2, v_2) \in [0, T] \times R^2$.

The continuity of f and the Lipschitz condition (2) guarantee the existence and uniqueness of the solution $y : [\alpha, T] \to R$ of problem (1).

Assume that the delay function g satisfied the condition $g(t) \le t, t \in [\alpha, 0]$ and the jump discontinuities to be known for sufficiently high-order derivatives of y and are given in the form:

$$\Delta : \xi_0 < \xi_1 < ... < \xi_M \tag{3}$$

We will construct a nonpolynomial spline function $s : [0, T] \to R$ in such a way that on each interval $[\xi_k, \xi_{k+1}]$ s be a nonpolynomial spline function.

We will use a collocation method of the order $O(h^{\beta+r+m})$ in $f^{(q)}, q = 0, 1, ..., r+1$. The function $f^{(q)}, q = 0, 1, ...$ is a function of the variables $t, y(t)$ and $y(g(t))$ and it will be obtained from the following algorithm:

If we denote $f^{(0)} := f\left(t, y(t), y(g(t))\right)$, then for all $q = 0, 1, 2, ...$

$$y^{(q+1)} := \frac{d^q f}{dt^q} := f^{(q)} := \frac{\partial^{(q-1)} f}{\partial t} + \frac{\partial^{(q-1)} f}{\partial y(t)} \cdot f + \frac{\partial^{(q-1)} f}{\partial y(g(t))} \cdot \frac{\partial y(g(t))}{\partial g(t)} \cdot \frac{\partial g(t)}{dt} \tag{4}$$

can be used as a recurrence formula. Let us consider the first interval $[\xi_0, \xi_1]$ which is $[0, \xi_1]$ and the uniform partition of this interval

$$\xi_0 = t_0 \le t_1 \le ... \le t_{m-j+1} \le ... \le t_m \le \xi_1 \tag{5}$$

Choosing a sufficiently large arbitrary positive integer m, let us define the nonpolynomial spline functions s, which approximate the solution y of (1)

$$s(t) := s_k^{[m]}(t) + \int_{t_k}^{t} f[t_1, s_k^{[m-1]}(t_1), s_k^{[m-1]}(g(t_1))]dt_1 \tag{6}$$

on the subinterval $t_k \le t \le t_{k+1}, (k = 1, 2, ..., n-1)$ such that
$s_{-1}^{[m]}(t_0) = \varphi(t_0), s_0(g(t)) = \varphi(g(t))$, and $s_0(t) = \varphi\left(t\right), t \in [\alpha, 0]$

Define the nonpolynomial spline function s_0 approximating the solution y of (1), on the first interval $I_1 : [t_0, \xi_1]$ by

$$s_0\left(t\right) = \varphi\left(t_1\right) + \int_{t_0}^{t} f(t_1, \varphi^{[j-1]}\left(t_1\right), \varphi^{[j-1]}(g(t_1)))dt_1 \tag{7}$$

Associating the following m iteration processes

$$s_1(t) = s_0^{[m]}(t_1) + \sum_{j=0}^{r} \frac{(t-t_1)^{j+1}}{(j+1)!} f^{(j)}(t_1, s_0^{[m]}(t_1), s_0^{[m]}(g(t_1))) \tag{8}$$

Let us denote the nonpolynomial spline function by $s_k, s_k \in S_m$, which is approximating the solution on the interval $I_k : [\xi_{k-1}, \xi_k]$. s_k is a nonpolynomial spline function such that $s_k : [\xi_{k-1}, \xi_k] \to R$. In a similar manner, following the introduced procedure one can easily construct spline functions on each subinterval $I_k : [\xi_{k-1}, \xi_k]$.

$$\begin{aligned} s_k(t) &= s_{k-1}(t_k) + \\ &+ \int_{t_{k-1}}^{t} f(t_{m-j+1}, s^{[j-1]}(t_{m-j+1}), s^{[j-1]}(g(t_{m-j+1})))dt_{m-j+1} \end{aligned} \tag{9}$$

and

$$s_k(t) = s_{k-1}^{[m]}(t) + \sum_{j=0}^{r} \frac{(t-t_k)^{j+1}}{(j+1)!} f^{(j)}(t_k, s_{k-1}^{[m]}(t_k), s_{k-1}^{[m]}(g(t_k))), \tag{10}$$

for $(k = 0, 1, ..., n-1)$.

We call the space $S_m(\Delta) = \{s$:there exists polynomials, $s_0, s_1, ..., s_n$ such that $s(x) = s_i(x)$ for $x \in I_i, (i = 1, 2, ...)$ and $D^j s_{i-1}(x_i) = D^j s_i(x_i)$ for $j = 0, 1, 2, ...\}$

Here the derivatives $s^{(j)}$ are left-hand limits of the segment of s defined on $[t_{k-1}, t_k]$.

This procedure yields a spline function $s \in S_m$ over the entire interval $[\xi_j, \xi_{j+1}]$ with the knots $\{t_k\}_{k=1}^{N}$.

By construction its obvious that $s \in C^r([\xi_j, \xi_{j+1}], R)$. Thus the exact solution of problem (1) can be written in the following form on the interval I_1 :

$$y(t) := y_k^{[m]}(t_k) + \int_{t_0}^{t} f(t_1, y^{[m-1]}(t_1), s^{[m-1]}(g(t_1)))dt_1 \tag{11}$$

where the following m-iterations processes are considered

$$y(t) := y^{[m]}(t_k) + \sum_{j=1}^{r} \frac{(t-t_k)^j}{j!} y^{(j)}(t_k) + \frac{y^{(r+1)}(\eta_k)}{(r+1)!}(t-t_k)^{r+1} \tag{12}$$

$$\begin{aligned} y(t) \quad : \quad &= y^{[j]}(t_k) + \\ &+ \int_{t_k}^{t} f(t_{m-j+1}, y^{[j-1]}(t_{m-j+1}), s^{[j-1]}(g(t_{m-j+1})))dt_{m-j+1} \end{aligned} \tag{13}$$

such that $j = 1, 2, ...m, t_k \leq \eta_k \leq t_{k+1}$ and $k = 0, 1, 2, ...$

It is then clear that the continuity of f and the Lipschitz condition (2) guarantee the existence and uniqueness of the solution (1) on every subinterval $[t_k, t_{k+1}]$.

2. Error estimation and Convergence

We show here that the global error of introduced numerical solution method for delay differential equations is bounded on the whole interval.

Theorem 1. *Assume that f satisfies the Lipschitz condition (2) and $s(t)$ is the nonpolynomial spline approximation of the solution of (1). Then the order of the introduced method is $O(h^{\beta+r+m})$.*

Proof. Let denote $L = \max\{L_1, L_2, ..., L_m\}$ as a Lipschitz constant and consider the first interval $I_1 : [\xi_0, \xi_1]$. For the error estimation of the introduced method using Lipschitz condition we get

$$\begin{aligned} |y(t) - s(t)| \quad &\leq \quad L \int_{t_0}^{t} \{ \left| y^{[m-1]}(t_1) - s^{[m-1]}(t_1) \right| + \\ &\quad + \left| y^{[m-1]}(g(t_1)) - s^{[m-1]}(g(t_1)) \right| \} dt_1 \\ &\leq \quad 2L^2 \int_{t_0}^{t} \int_{t_0}^{t_1} \{ \left| y^{[m-2]}(t_2) - s^{[m-2]}(t_2) \right| + \\ &\quad + \left| y^{[m-2]}(g(t_2)) - s^{[m-2]}(g(t_2)) \right| \} dt_2 dt_1 \\ &\quad ... \\ &\leq \quad 2^{m-1} L^m \int_{t_0}^{t} \int_{t_0}^{t_1} ... \int_{t_0}^{t_{m-1}} \{ \left| y^{[0]}(t_m) - s^{[0]}(t_m) \right| + \\ &\quad + \left| y^{[0]}(g(t_m)) - s^{[0]}(g(t_m)) \right| \} dt_m ... dt_1 \end{aligned} \tag{14}$$

and consequently we obtain

$$|y(t) - s(t)| \leq \frac{2^m L^m}{(r+m+1)!} h^{r+m+1} \cdot W(h) = O(h^{r+m+1}) \tag{15}$$

where $W(h) = \max\left\{\omega(y^{(r+1)}(t), h)\right\}$ such that $\omega(y^{(r+1)}, h)$ is the continuity moduli of the function $y^{(r+1)}$ and h is the step-length..

Similarly the difference $|y\acute{}(t) - s\acute{}(t)|$ can be estimated easily. Thus

$$|y\acute{}(t) - s\acute{}(t)| \leq L[\left|y^{[m-1]}(t) - s^{[m-1]}(t)\right| + \left|y^{[m-1]}(g(t)) - s^{[m-1]}(g(t))\right|]$$

$$\leq 2L^2 \int_{t_0}^{t} [\left|y^{[m-2]}(t_2) - s^{[m-2]}(t_2)\right| + \left|y^{[m-2]}(g(t_2)) - s^{[m-2]}(g(t_2))\right|] dt_2$$

$$\leq 2^{m-1} L^m \int_{t_0}^{t} \int_{t_0}^{t_2} \ldots \int_{t_0}^{t_{m-1}} [\left|y^{[0]}(t_m) - s^{[0]}(t_m)\right| +$$

$$+ \left|y^{[0]}(g(t_m)) - s^{[0]}(g(t_m))\right|] dt_m dt_{m-1} \ldots dt_2$$

and finally we obtain

$$|y\acute{}(t) - s\acute{}(t)| \leq \frac{2^m L^m}{(r+m)!} h^{r+m} \cdot W(h) = O(h^{\beta+r+m}) \tag{16}$$

Hence choosing for $q = 2, 3, ..., r+1$ we get

$$\left|y^{(q)}(t) - s^{(q)}(t)\right| \leq \frac{2^k L^k}{(r+m)!} h^{r+m} \cdot W(h) = O(h^{\beta+r+m}) \tag{17}$$

This completes the proof of the theorem. Let denote the difference

$$e(t) = |y(t) - s(t)|, e\acute{}(t) = |y\acute{}(t) - s\acute{}(t)|. \tag{18}$$

Lemma 2. *Suppose $f \in C^r([0,T] \times [\alpha,T] \times [\alpha,T], R), r \in N$ and f satisfy the Lipschitz condition (2) with a constant $L = \max\{L_1, L_2, ..., L_m\}$, then there exist constants C_1 and C_2 which are independent of h such that*

$$e(t) \leq C_1 h^{r+m} W(h) = O(h^{\beta+r+m})$$

$$e\acute{}(t) \leq C_2 h^{r+m} W(h) = O(h^{\beta+r+m}) \tag{19}$$

where $0 < \beta \leq 1$. Similarly it can be easily shown that there exists a constant C_3 which is independent of h such that the following inequality holds

$$\left| y^{(q)}(t) - s^{(q)}(t) \right| \leq C_3 h^{r+m} W(h) = O(h^{\beta+r+m}) \tag{20}$$

where $q = 2, 3, ..., r+1$ and $t \in [t_k, t_{k+1}]$. Proof of Lemma is obvious.

From (20) and defined procedure we have the following subsequent assertion:

Theorem 3. *Let $y : [0,T] \to R$ be the exact solution of the problem (1). If $s : [0,T] \to R$ is the nonpolynomial spline approximation of the solution of (1), defined by the introduced procedure, then the following inequalities hold:*

$$|y(t) - s(t)| \leq \frac{C_0}{(r+m+1)!} h^{r+m+1} W(h) \tag{21}$$

$$\left| y^{(q)}(t) - s^{(q)}(t) \right| \leq \frac{C_0}{(r+m)!} h^{r+m} W(h) \tag{22}$$

where $C_0 = 2^m L^m, q = 0, 1, 2, ..., r+1$, for all $t \in [t_k, t_{k+1}]$ and consequently

$$\left| y^{(q)}(t) - s^{(q)}(t) \right| \leq C h^{r+m} W(h) \tag{23}$$

holds for all $t \in [t_k, t_{k+1}]$ and for $q = 0, 1, 2, ..., r+1$. Here C is a constant independent of h.

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