STUDIA UNIV. "BABEŞ–BOLYAI", INFORMATICA, Volume **XLIV**, Number 2, 1999

A GENERAL CLASS OF NONPRODUCT QUADRATURE FORMULAS

PETRU P. BLAGA

ABSTRACT. **Abstract.** A general class of fifth degree approximate integration formulas for hypercubes is constructed. If the integrand is a real function of n independent real variables, then a $2^n + \binom{n}{k}2^k + 1$ point class nonproduct quadraturae is obtained. A lot of known multiple quadrature formulas are included in this class. In the particular cases $n = 2, 3$, comparative numerical examples are considered.

1. Introduction

Here we consider an approximate evaluation to the multiple definite integral

$$I_n[f] = \int_{-1}^{1} \cdots \int_{-1}^{1} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n. \tag{1}$$

A very wide class of $2^n + \binom{n}{k}2^k + 1$ point nonproduct quadrature rules of 5th degree is obtained. The evaluation points of integrated function f are symmetrically placed inside of the domain $S_n = [-1, 1]^n$ of the integral $I_n[f]$. As such the rule is required to be exact for the monomials of degree 0, 2, 4. Constructed approximate integration formulas extend the 2^n+2n+1 point quadrature presented in [2] and $2^n + 2^{n-1}n + 1$ point quadrature considered in [3], and also that of Das and Pradham [5] (see also Blaga [1]). For $n = 2$ and $n = 3$ other known multiple quadrature formulas are obtained: Blaga [1],[2],[3], Burnside [4] (see also Stroud [9], p. 248), Hammer and Stroud [8] (see also Stroud [9], p. 231), Mustard, Lyness, Blatt [7], and product Gauss formula (see Stroud [9], p. 249).

2. Fifth Degree Integration Formula

Let us take the following $N = 2^n + \binom{n}{k}2^k + 1$ points: $(\beta_1, \ldots, \beta_n)$, $(\gamma_1, \ldots, \gamma_n)$ and $(0, \ldots, 0)$, where each β_i $(1 \leqslant i \leqslant n)$ is either $-\lambda\alpha$ or $+\lambda\alpha$, i.e. the corners of the hypercube $[-\lambda\alpha, +\lambda\alpha]^n$, and $n-k$ (and only $n-k$) of the γ_i $(1 \leqslant i \leqslant n)$ are zero and all the others k of γ_i equal either $-\alpha$ or α $(1 \leqslant k < n)$. On the one hand we have that the number of $(\beta_1, \ldots, \beta_n)$ type points is 2^n, on the other hand the number of $(\gamma_1, \ldots, \gamma_n)$ type points is $\binom{n}{k}2^k$, and the last type of points

1991 *Mathematics Subject Classification.* 65D32.

are situated at the corners of hypercubes $[-\alpha,\alpha]^k$ from the hyperplanes given by $x_{j_s} = 0$ $(s = \overline{1, n-k})$, $1 \leqslant j_1 < \cdots < j_{n-k} \leqslant n$, respectively. Taking into account that the center of the hypercube S_n is also considered it results that $N = 2^n + \binom{n}{k} 2^k + 1$. In order to all evaluation points belong to the hypercube S_n the real positive parameters λ and α must satisfy the conditions $\alpha < 1$ and $\lambda\alpha \leqslant 1$.

We shall construct an approximate rule to $I_n[f]$ of the following type

$$\begin{aligned} Q_{n,k}[f] = A_0 f(0,\dots,0) + A_1 \sum\nolimits_1 f(\gamma_1,\dots,\gamma_n) \\ + A_2 \sum\nolimits_2 f(\beta_1,\dots,\beta_n). \end{aligned} \tag{2}$$

As we have seen, in the formula (2) the the first sum, $\sum_1$, has $\binom{n}{k} 2^k$ terms, and the second sum, $\sum_2$, has 2^n terms. Such that the quadrature (2) will be a $2^n + \binom{n}{k} 2^k + 1$ point (generally) nonproduct formula.

The coefficients A_0, A_1, A_2 and the parameters α and λ will be determined such as to make the rule exact for all monomials of degree less or equal to five, i.e.

$$Q_n[f] = I_n[f], \tag{3}$$

for

$$f = x_1^{k_1} \dots x_n^{k_n}, \quad \text{where} \quad 0 \leqslant k_1 + \cdots + k_n \leqslant 5. \tag{4}$$

We remark the exactness of the formula (3) for all monomials (4) containing at least one odd power k_i. On the other hand, taking into account that the formula (2) has the evaluation points of the function f symmetrically situated over the integration domain S_n (if $\alpha \in (0,1)$ and $\lambda\alpha \in (0,1]$), we have to require that (3) to be exact only for the monomials

$$f = 1,\, x_1^2,\, x_1^4, x_1^2 x_2^2, \tag{5}$$

to obtain a fifth degree exactness quadrature formula.

The exactness conditions of the formula (3) for the monomials (5) give the following nonlinear algebraic system in A_0, A_1, A_2 and α, λ:

$$\begin{cases} A_0 & + \binom{n}{k} 2^k A_1 & + 2^n A_2 & = 2^n \\ & \binom{n-1}{k-1} 2^k \alpha^2 A_1 & + 2^n \lambda^2 \alpha^2 A_2 & = \frac{1}{3} 2^n \\ & \binom{n-1}{k-1} 2^k \alpha^4 A_1 & + 2^n \lambda^4 \alpha^4 A_2 & = \frac{1}{5} 2^n \\ & \binom{n-2}{k-2} 2^k \alpha^4 A_1 & + 2^n \lambda^4 \alpha^4 A_2 & = \frac{1}{9} 2^n, \end{cases} \tag{6}$$

with $\binom{n-2}{-1} = 0$.

To obtain the solution of system (6), we must remark the special case $5n - 9k + 4 = 0$, i.e. $(n,k) \in \left\{ (9\ell+1, 5\ell+1) \;\middle|\; \ell = 1, 2, \dots \right\}$. In this case, from the last two equations of (6), it is obtained that $A_2 = 0$, and the system (6) is equivalent with

$$\left\{\begin{array}{lll} A_0 & +\binom{n}{k}2^k A_1 & = 2^n \\ & \binom{n-1}{k-1}2^k\alpha^2 A_1 & = \frac{1}{3}2^n \\ & \binom{n-1}{k-1}2^k\alpha^4 A_1 & = \frac{1}{5}2^n. \end{array}\right. \tag{7}$$

From the last two equations of (7) results that $\alpha^2 = \frac{3}{5}$, and then

$$A_1 = \frac{5}{9}\frac{2^{n-k}}{\binom{n-1}{k-1}} \quad \text{and} \quad A_0 = \frac{2^{n+2}}{9k}\cdot$$

So, in this case we obtain the following $\binom{n}{k}2^k + 1$ point 5th degree formula

$$\begin{aligned} Q_{n,k}[f] =& \frac{2^{n-k}}{9}\Bigg[\frac{2^{k+2}}{k} f(0,\ldots,0) \\ & + \frac{5}{\binom{n-1}{k-1}} \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\Bigg(0,\ldots,\underset{\substack{\uparrow \\ i_1}}{\pm\sqrt{\frac{3}{5}}},\ldots,\underset{\substack{\uparrow \\ i_k}}{\pm\sqrt{\frac{3}{5}}},\ldots,0\Bigg)\Bigg]. \end{aligned} \tag{8}$$

One remarks that the smallest n dimensional formula is obtained for $n = 10$ ($k = 6$), and it is a 13341 point 5th degree formula.

In the following we consider that $5n - 9k + 4 \neq 0$.

From the fourth and third equations of (6) we get

$$A_2 = \frac{5n - 9k + 4}{45\,(n-k)\,\lambda^4\alpha^4},$$

and then taking into account the second equation of (6) we obtain the following equivalent relations between the parameters λ and α:

$$\alpha^2 = \frac{4\,(n-1)\,\lambda^2 + 5n - 9k + 4}{15\,(n-k)\,\lambda^2}, \quad \lambda^2 = \frac{5n - 9k + 4}{15\,(n-k)\,\alpha^2 - 4\,(n-1)}\cdot$$

Finally, we get the solution of the system (6)

$$\begin{aligned} A_0 &= 2^n \frac{5k\,(n-k)\left(9\lambda^4\alpha^4 - 1\right) - 4n\,(n-1)\,\lambda^4 + 4k\,(k-1)}{45k\,(n-k)\,\lambda^4\alpha^4} \\ &= -2^{n+2}\frac{45k\,(k-1)\,\alpha^4 - 30k\,(n-1)\,\alpha^2 + (n-1)\,(5n+4)}{45k\,(5n-9k+4)\,\alpha^4}, \\ A_1 &= \frac{2^{n-k+2}}{45\binom{n-2}{k-1}\alpha^4}, \\ A_2 &= \frac{\left[15\,(n-k)\,\alpha^2 - 4\,(n-1)\right]^2}{45\,(n-k)\,(5n-9k+4)\,\alpha^4}\cdot \end{aligned} \tag{9}$$

Thus we have obtained the following quadrature rule

$$Q_{n,k}[f] = \frac{1}{45\alpha^4}\Bigg[-2^{n+2}\frac{45k(k-1)\alpha^4 - 30k(n-1)\alpha^2 + (n-1)(5n+4)}{k(5n-9k+4)} \times f(0,\ldots,0) + \frac{2^{n-k+2}}{\binom{n-2}{k-1}}\sum\nolimits_1 f(\gamma_1,\ldots,\gamma_n) + \frac{\left[15(n-k)\alpha^2 - 4(n-1)\right]^2}{(n-k)(5n-9k+4)}\sum\nolimits_2 f(\beta_1,\ldots,\beta_n)\Bigg]. \tag{10}$$

In the following table we resume the conditions for α^2 and λ^2 such that all evaluation points in the quadrature formula (10) are placed inside of the domain S_n.

	α^2	λ^2
$5n+4>9k$	$\left[\frac{2(n-1)}{5n-3k-2};1\right)$	$\left(\frac{5n-9k+4}{11n-15k+4};\frac{5n-3k-2}{2(n-1)}\right]$
$5n+4<9k$	$\left(0;\frac{2(n-1)}{5n-3k-2}\right]$	$\left(\frac{9k-5n-4}{4(n-1)};\frac{5n-3k-2}{2(n-1)}\right]$

For the first two values of n ($n=2, n=3$), the coefficients of quadrature formula (10) and the relation between the parameters α and λ are given in the following table

	A_0	A_1	A_2	
$n=2,\ k=1$	$\frac{32\left(15\alpha^2-7\right)}{225\alpha^4}$	$\frac{8}{45\alpha^4}$	$\frac{\left(15\alpha^2-4\right)^2}{225\alpha^4}$	$\alpha^2=\frac{4\lambda^2+5}{15\lambda^2}$
$n=3,\ k=1$	$\frac{32\left(30\alpha^2-19\right)}{225\alpha^4}$	$\frac{16}{45\alpha^4}$	$\frac{\left(15\alpha^2-4\right)^2}{225\alpha^4}$	$\alpha^2=\frac{4\lambda^2+5}{15\lambda^2}$
$n=3,\ k=2$	$\frac{32\left[1-5\left(3\alpha^2-2\right)^2\right]}{45\alpha^4}$	$\frac{8}{45\alpha^4}$	$\frac{\left(15\alpha^2-8\right)^2}{45\alpha^4}$	$\alpha^2=\frac{8\lambda^2+1}{15\lambda^2}$

3. Particular Cases

1. *Generalized Das-Pradham quadrature formula* [5] (see also [1],[2] and [3]). This quadrature is obtained considering $\alpha^2 = \frac{2(n-1)}{5n-3k-2}$ or equivalently $\lambda^2 = \frac{5n-3k-2}{2(n-1)}$

$(\lambda\alpha=1)$. In this case from (9) we get

$$A_0 = -2^n \frac{5n\,(5n-9k-4)+4\,(9k+1)}{45k\,(n-1)},$$
$$A_1 = 2^{n-k} \frac{(5n-3k-2)^2}{45\,(n-1)^2 \binom{n-2}{k-1}},$$
$$A_2 = \frac{5n-9k+4}{45\,(n-k)},$$

and the corresponding quadrature formula is

$$\begin{aligned} Q^{[1]}_{n,k}[f]) = \frac{1}{45}\Bigg[&- 2^n \frac{5n\,(5n-9k+4)+4\,(9k+1)}{k\,(n-1)} f(0,\ldots,0) \\ &+ 2^{n-k} \frac{(5n-3k-2)^2}{(n-1)^2 \binom{n-2}{k-1}} \sum\nolimits_1 f(\gamma_1,\ldots,\gamma_n) \\ &+ \frac{5n-9k+4}{n-k} \sum\nolimits_2 f(\beta_1,\ldots,\beta_n) \Bigg]. \end{aligned} \tag{11}$$

This formula has been constructed in [1].

We also remark that formula (11) gives the Das–Pradham quadrature formula (see [5] and [3]), in the case $k = n-1$, and Mustard–Lyness–Blatt quadrature (see [7] and [2]), in the case $k = 1$.

2. *Quadrature formula with the same coordinates of evaluation points*. This quadrature is obtained considering $\lambda^2 = 1$ or equivalently $\alpha^2 = \frac{3}{5}$. In this case from (9) we get

$$A_0 = 2^{n+2}\frac{5+9k-5n}{81k}, \quad A_1 = \frac{5}{81}\frac{2^{n-k+2}}{\binom{n-2}{k-1}}, \quad A_2 = \frac{5}{81}\frac{5n-9k+4}{n-k},$$

and the corresponding quadrature formula is

$$\begin{aligned} Q^{[2]}_{n,k}[f] = \frac{1}{81}\Bigg[&2^{n+2}\frac{5+9k-5n}{k} f(0,\ldots,0) \\ &+ 2^{n-k+2}\frac{5}{\binom{n-2}{k-1}} \sum\nolimits_1 f(\gamma_1,\ldots,\gamma_n) \\ &+ \frac{5\,(5n-9k+4)}{n-k} \sum\nolimits_2 f(\beta_1,\ldots,\beta_n) \Bigg]. \end{aligned} \tag{12}$$

Cases $k = 1$ and $k = n-1$ are presented in [2] and [3] respectively. In the case $n = 2$ the product Gauss quadrature formula is obtained (see [9], p.249) which is also a particular case of Hammer and Stroud quadrature formula (see [6] and [9], p. 231).

Here is also included the quadrature formula (8) $(5n-9k+4=0)$.

3. *Quadrature formula with* $A_0 = 0$. This quadrature is a nonproduct $\left[2^k + \binom{n}{k}\right] 2^{n-k}$ point of 5th exactness degree formula. From the condition $A_0 = 0$, it results the algebraic equation

$$45k\,(k-1)\,\alpha^4 - 30k\,(n-1)\,\alpha^2 + (n-1)\,(5n+4) = 0. \tag{13}$$

It must be considered the following two cases:

Case $k = 1$, when $\alpha^2 = \frac{5n+4}{30}$ or equivalently $\lambda^2 = \frac{10}{5n-4}$ ($\lambda^2\alpha^2 = \frac{5n+4}{3(5n-4)}$). One remarks that only for $2 \leqslant n \leqslant 5$ all evaluation points are inside of S_n. Using formulas (9) we have

$$A_1 = 2^{n+3}\frac{5}{(5n+4)^2}, \quad A_2 = \left(\frac{5n-4}{5n+4}\right)^2.$$

The corresponding quadrature has been given in [2].

In the case $1 < k < n$, the equation (13) with the unknown α^2 has real solutions only for $5n - 9k + 4 > 0$, i.e. $2 \leq k \leq \left[\frac{5n+4}{9}\right]$, and necessarily $n > 2$.

Using results of the two cases and the formulas (9), in Table 1 we give the admissible solutions and the corresponding elements of $Q_{n,k}^{[2]}[f]$ for $n = \overline{2,9}$.

Cases $n = 2$ and $n = 3$ with $k = 1$ lead to the Burnside quadrature formula [4] (see also [8], [9] p. 233 and p. 248, and [2], [3]), and respectively the Hammer–Stroud quadrature formula [6] (see also [9], p.263 and [2]).

From the Table 1, we observe there are two formulas when $n = 8, k = 4$ and $n = 9, k = 5$, and there not exists any quadrature when $n = 9, k = 2$.

One also remarks that in the two cases the coefficients A_1 and A_2 are positive.

4. *Quadrature with positive coefficients*. We observe that all the time the coefficient A_1 is positive. Consequently, in order to obtain quadrature formulas of the type (10) with positive coefficients it must that $A_0 > 0$ and $A_2 > 0$.

Again, it must be considered the following two cases:

Case $k = 1$, when $\alpha^2 > \frac{5n+4}{30}$. One remarks that only for $2 \leqslant n \leqslant 5$ all evaluation points are inside of S_n. Using formulas (9) we have

$$A_0 = 2^{n+2}\frac{30\alpha^2 - 5n - 4}{225\alpha^4} > 0, \quad A_1 = 2^{n+1}\frac{1}{45\alpha^4} > 0, \quad A_2 = \frac{1}{\lambda^4\alpha^4} > 0.$$

Such a quadrature has been obtained in [2].

In the case $1 < k < n$, from $A_2 > 0$ and $A_0 > 0$, it must be determined $\alpha^2 \in \left(\frac{2(n-1)}{5n-3k-2}; 1\right)$ satisfying the inequality

$$45k\,(k-1)\,\alpha^4 - 30k\,(n-1)\,\alpha^2 + (n-1)\,(5n+4) < 0.$$

Using results of the two cases, in Table 2 we give the admissible intervals of parameter α^2 for $n = \overline{2,9}$.

4. Error Analysis

If the integrated function $f \in C^6(S_n)$, then the Taylor's formula is valid:

$$f(x) = \sum_{i=0}^{5} \frac{1}{i!}\left(x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}\right)^{(i)} f(0) + \frac{1}{6!}\left(x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}\right)^{(6)} f(\xi), \ \xi \in S_n.$$

Taking into account the error

$$R_{n,k}[f] = I_n[f] - Q_{n,k}[f] = 0,$$

for all monomials $f(x) = x_1^{k_1} \ldots x_n^{k_n}$, $0 \leqslant k_1 + \cdots + k_n \leqslant 5$, it results that

$$R_{n,k}[f] = \frac{1}{6!} R_{n,k}\left[\left(x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}\right)^{(6)} f(\xi)\right].$$

Moreover, having the evaluation points symmetrically situated inside of S_n, we obtain that

$$R_{n,k}[f] = \frac{1}{6!}\left\{R_{n,k}\left[x_1^6\right] \sum_{i=1}^{n} \frac{\partial^6 f(\xi)}{\partial x_i^6} + 15 R_{n,k}\left[x_1^4 x_2^2\right] \sum_{i \neq j} \frac{\partial^6 f(\xi)}{\partial x_i^4 \partial x_j^2} + 90 R_{n,k}\left[x_1^2 x_2^2 x_3^2\right] \sum_{i<j<\ell} \frac{\partial^6 f(\xi)}{\partial x_i^2 \partial x_j^2 \partial x_\ell^2}\right\},$$

and consequently

$$|R_{n,k}[f]| \leqslant \frac{1}{6!}\left\{L\left|R_{n,k}\left[x_1^6\right]\right| + 15M\left|R_{n,k}\left[x_1^4 x_2^2\right]\right| + 90N\left|R_{n,k}\left[x_1^2 x_2^2 x_3^2\right]\right|\right\},$$

where

$$L = \sup_{x \in S_n}\left|\sum_{i=1}^{n} \frac{\partial^6 f(x)}{\partial x_i^6}\right|, \qquad M = \sup_{x \in S_n}\left|\sum_{i \neq j} \frac{\partial^6 f(x)}{\partial x_i^4 \partial x_j^2}\right|,$$

$$N = \sup_{x \in S_n}\left|\sum_{i<j<\ell} \frac{\partial^6 f(x)}{\partial x_i^2 \partial x_j^2 \partial x_\ell^2}\right|.$$

It must observe the last term in these formulas is zero when $n = 2$. The general error bound of quadrature in this case is given by:

$$\begin{aligned} |R_{2,1}[f]| &\leqslant \frac{1}{180}\left(\left|\frac{1}{7} - \frac{4}{45}\alpha^2 - \frac{1}{9}\lambda^2\alpha^2\right| L + \left|1 - \frac{5}{3}\lambda^2\alpha^2\right| M\right) \\ &= \frac{1}{135\left|15\alpha^2 - 4\right|}\left(\frac{1}{35}\left|35\alpha^4 - 51\alpha^2 + 15\right| L + \left|5\alpha^2 - 3\right| M\right), \end{aligned}$$

and the corresponding error bounds for the quadrature formulas considered in the previous section are:

$$\begin{aligned}
&\left|R^{[1]}_{2,1}[f]\right| \leqslant \frac{1}{270}\left(\frac{1}{175}L+M\right), && \left(\alpha^2=\frac{2}{5},\ \lambda^2\alpha^2=1,\ \text{see [5]}\right),\\
&\left|R^{[2]}_{2,1}[f]\right| \leqslant \frac{1}{7875}L, && \left(\alpha^2=\frac{3}{5},\ \lambda^2\alpha^2=\frac{3}{5}\right)\\
&\left|R^{[3]}_{2,1}[f]\right| \leqslant \frac{1}{1215}\left(\frac{53}{525}L+2M\right), && \left(\alpha^2=\frac{7}{15},\ \lambda^2\alpha^2=\frac{7}{9}\right),\\
&\left|R^{[4]}_{2,1}[f]\right| \leqslant \frac{1}{2430}\left(\frac{31}{105}L+M\right), && \left(\alpha^2=\frac{2}{3},\ \lambda^2\alpha^2=\frac{5}{9}\right).
\end{aligned}$$

We also consider the error bounds for $n=3$, ($k=1$ and $k=2$).

In the case $k=1$ we have

$$\begin{aligned}
|R_{3,1}[f]| \leqslant \frac{1}{90}\Bigg(\left|\frac{1}{7}-\frac{4}{45}\alpha^2-\frac{1}{9}\lambda^2\alpha^2\right|L+\left|1-\frac{4}{3}\alpha^2-\frac{5}{3}\lambda^2\alpha^2\right|M&\\
+10\left|\frac{1}{3}-\lambda^2\alpha^2\right|N\Bigg)&,
\end{aligned}$$

and in the particular cases from the previous section we get

$$\begin{aligned}
&\left|R^{[1]}_{3,1}[f]\right| \leqslant \frac{1}{45}\left(\frac{1}{525}L+\frac{3}{5}M+\frac{10}{3}N\right), && \left(\alpha^2=\frac{2}{5},\ \lambda^2\alpha^2=1\right),\\
&\left|R^{[2]}_{3,1}[f]\right| \leqslant \frac{2}{225}\left(\frac{1}{35}L+M+\frac{10}{3}N\right), && \left(\alpha^2=\frac{3}{5},\ \lambda^2\alpha^2=\frac{3}{5}\right),\\
&\left|R^{[3]}_{3,1}[f]\right| \leqslant \frac{1}{1485}\left(\frac{587}{1575}L+\frac{199}{15}M+40N\right), && \left(\alpha^2=\frac{19}{30},\ \lambda^2\alpha^2=\frac{19}{33}\right),\\
&\left|R^{[4]}_{3,1}[f]\right| \leqslant \frac{1}{405}\left(\frac{31}{315}L+\frac{11}{3}M+10N\right), && \left(\alpha^2=\frac{2}{3},\ \lambda^2\alpha^2=\frac{5}{9}\right).
\end{aligned}$$

In the case $k=2$ we have:

$$\begin{aligned}
|R_{3,2}[f]| \leqslant \frac{1}{90}\Bigg(\left|\frac{1}{7}-\frac{8}{45}\alpha^2-\frac{1}{45}\lambda^2\alpha^2\right|L+\left|1-\frac{4}{3}\alpha^2-\frac{1}{3}\lambda^2\alpha^2\right|M&\\
+10\left|\frac{1}{3}-\frac{1}{5}\lambda^2\alpha^2\right|N\Bigg)&,
\end{aligned}$$

and in the particular cases from the previous section we get

$$\left|R_{3,2}^{[1]}[f]\right| \leqslant \frac{1}{135}\left(\frac{1}{35}L+\frac{1}{7}M+2N\right), \qquad \left(\alpha^2=\frac{4}{7},\ \lambda^2\alpha^2=1\right),$$

$$\left|R_{3,2}^{[2]}[f]\right| \leqslant \frac{2}{1125}\left(\frac{1}{7}L+\frac{40}{3}N\right), \qquad \left(\alpha^2=\frac{3}{5},\ \lambda^2\alpha^2=\frac{3}{5}\right),$$

$$\left|R_{3,2}^{[3]}[f]\right| \leqslant \frac{2}{675}\left(\frac{28\sqrt{5}-55}{315}L+\frac{3\sqrt{5}-5}{3}M+4\left(5-\sqrt{5}\right)N\right),$$

$$\left(\alpha^2=\frac{10+\sqrt{5}}{15},\ \lambda^2\alpha^2=\frac{8\sqrt{5}-15}{15}\right),$$

$$\left|R_{3,2}^{[4]}[f]\right| \leqslant \frac{4}{135}\left(\frac{2}{315}L+M\right), \qquad \left(\alpha^2=\frac{2}{3},\ \lambda^2\alpha^2=\frac{1}{3}\right).$$

5. Numerical Examples

To compare the quadrature rules obtained in the previous sections, we have considered the four quadratures when $n=2$ ($k=1$) and $n=3$ ($k=1$ and $k=2$).

The values of parameters α, λ and the coefficients A_0, A_1, A_2 are given in Tables 3–5.

Table 6, Table 7 and Table 8 contain the exact and approximate values obtained by the four quadrature formulas, and also the error bounds, for the following four integrated functions:

(1) $f(x,y)=\frac{1}{(3+x+y)^2}$, with $I_2[f]=\ln\frac{9}{5}=\ln 1.8$,
(2) $f(x,y)=e^{xy}$, with $I_2[f]$ computed by series expansion,
(3) $f(x,y)=\sqrt{2+x+y}$, with $I_2[f]=\frac{32}{15}\left(4-\sqrt{2}\right)$,
(4) $f(x,y)=\frac{1}{\sqrt{3+x+y}}$, with $I_2[f]=\frac{4}{3}\left(5\sqrt{5}-6\sqrt{3}+1\right)$,

respectively

(1) $f(x,y,z)=\frac{1}{(4+x+y+z)^3}$, with $I_3[f]=\frac{1}{2}\ln\frac{189}{125}$,
(2) $f(x,y,z)=e^{xyz}$, with $I_3[f]$ computed by series expansion,
(3) $f(x,y,z)=\sqrt{3+x+y+z}$, with $I_3[f]=\frac{64}{35}\left(9\sqrt{6}-16+\sqrt{2}\right)$,
(4) $f(x,y,z)=\frac{1}{\sqrt{4+x+y+z}}$, with $I_3[f]=\frac{8}{15}\left(49\sqrt{7}-75\sqrt{5}+27\sqrt{3}-1\right)$.

References

[1] Blaga, P. P., *An Approximate Formula for Multiple Integrals*, Facta Universitatis (Niš) (to appear).
[2] Blaga, P. P., *A Class of Multiple Nonproduct Quadrature Formulas*, in "Analysis, Functional Equations, Approximation and Convexity", Carpatica, Cluj–Napoca, 1999, pp. 32–39.
[3] Blaga, P. P., *A New Class of Multiple Nonproduct Quadrature Formulas*, to appear.
[4] Burnside, W., *An Approximate Quadrature Formula*, Messenger of Math. **37** (1908), 166–167.

[5] Das, R. N. and Pradham, G., *A Numerical Quadrature of Function of More Than One Real Variable*, Facta Universitatis (Niš) **11** (1996), 113–118.

[6] Hammer, P. C. and Stroud, A. H., *Numerical Evaluation of Multiple Integrals II*, Math. Tables Aids Comput. **12** (1958), 272–280.

[7] Mustard, D., Lyness, J. N. and Blatt, J. M., *Numerical Quadrature in N Dimensions*, Computer J. **6** (1963–1964), 75–87.

[8] Stroud, A. H., *Some Fifth Degree Integration Formulas for Symmetric Regions*, Math. of Comput. **20** (1966), 90–97.

[9] Stroud, A. H., *Approximate Calculation of Multiple Integrals*, Prentice–Hall, Englewood Cliffs, New Jersey, 1971.

Faculty of Mathematics and Informatics, "Babeş–Bolyai" University, 3400 Cluj–Napoca

E-mail address: blaga@math.ubbcluj.ro

		n=2	n=3	n=4	n=5
k=1	α^2	$\frac{7}{15}$	$\frac{19}{30}$	$\frac{4}{5}$	$\frac{29}{30}$
	α	0.68313	0.79582	0.89443	0.98319
	$\lambda\alpha$	0.88192	0.75879	0.70711	0.67847
	A_1	0.81633	0.88643	1.11111	1.52200
	A_2	0.18367	0.33518	0.44444	0.52438
k=2	α^2		$\frac{10+\sqrt{5}}{15}$	$\frac{5-\sqrt{5}}{5}$	$\frac{20-\sqrt{110}}{15}$
	α		0.90318	0.74350	0.79632
	$\lambda\alpha$		0.43883	0.85065	0.74595
	A_1		0.26716	0.58179	0.58947
	A_2		0.59926	0.12732	0.26316
k=3	α^2				$\frac{30+\sqrt{30}}{45}$
	α				0.88791
	$\lambda\alpha$				0.45395
	A_1				0.19068
	A_2				0.52329

		n=6	n=7	n=8	n=9
k=2	α^2	$\frac{5-2\sqrt{2}}{3}$	$\frac{10-\sqrt{35}}{5}$	$\frac{35-\sqrt{455}}{15}$	
	α	0.85080	0.90376	0.95461	
	$\lambda\alpha$	0.70305	0.67850	0.66230	
	A_1	0.67858	0.85273	1.14174	
	A_2	0.36383	0.44039	0.50049	
k=3	α^2	$\frac{15-\sqrt{21}}{18}$	$\frac{15-\sqrt{30}}{15}$	$\frac{105-\sqrt{1785}}{90}$	$\frac{60-2\sqrt{165}}{45}$
	α	0.76075	0.79678	0.83500	0.87317
	$\lambda\alpha$	0.81874	0.73528	0.69852	0.67680
	A_1	0.35384	0.35288	0.39008	0.46602
	A_2	0.11539	0.22808	0.31736	0.38835
k=4	α^2		$\frac{20+\sqrt{10}}{30}$	$\frac{35-\sqrt{70}}{45}$; $\frac{35+\sqrt{70}}{45}$	$\frac{40-\sqrt{130}}{45}$
	α		0.87868	0.76932; 0.98168	0.79719
	$\lambda\alpha$		0.46432	0.79396; 0.50845	0.72625
	A_1		0.11929	0.20301; 0.07657	0.20122
	A_2		0.47809	0.11185; 0.66501	0.20769
k=5	α^2				$\frac{10-\sqrt{2}}{15}$; $\frac{10+\sqrt{2}}{15}$
	α				0.75656; 0.87232
	$\lambda\alpha$				0.98850; 0.47210
	A_1				0.12403; 0.07018
	A_2				0.02327; 0.44736

TABLE 1. Elements of quadrature formulas $Q^{[3]}_{n,k}$, $n = \overline{2,9}$.

n	k=1	k=2	k=3	k=4	k=5
2	$\left(\frac{7}{15};1\right)$				
3	$\left(\frac{19}{30};1\right)$	$\left(\frac{10+\sqrt{5}}{15};1\right)$			
4	$\left(\frac{4}{5};1\right)$	$\left(\frac{5-\sqrt{5}}{5};1\right)$			
5	$\left(\frac{29}{30};1\right)$	$\left(\frac{20-\sqrt{110}}{15};1\right)$	$\left(\frac{30+\sqrt{30}}{45};1\right)$		
6		$\left(\frac{5-2\sqrt{2}}{3};1\right)$	$\left(\frac{15-\sqrt{21}}{18};1\right)$		
7		$\left(\frac{10-\sqrt{35}}{5};1\right)$	$\left(\frac{15-\sqrt{30}}{15};1\right)$	$\left(\frac{20+\sqrt{10}}{30};1\right)$	
8		$\left(\frac{35-\sqrt{455}}{15};1\right)$	$\left(\frac{105-\sqrt{1785}}{90};1\right)$	$\left(\frac{7}{13};\frac{35-\sqrt{70}}{45}\right) \cup \left(\frac{35+\sqrt{70}}{45};1\right)$	
9			$\left(\frac{60-2\sqrt{165}}{45};1\right)$	$\left(\frac{40-\sqrt{130}}{45};1\right)$	$\left(\frac{4}{7};\frac{10-\sqrt{2}}{15}\right) \cup \left(\frac{10+\sqrt{2}}{15};1\right)$

TABLE 2. Admissible values of parameter α^2, $n=\overline{2,9}$.

	$Q_{2,1}^{[1]}$	$Q_{2,1}^{[2]}$	$Q_{2,1}^{[3]}$	$Q_{2,1}^{[4]}$
α^2	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{7}{15}$	$\frac{2}{3}$
λ^2	$\frac{5}{2}$	1	$\frac{5}{3}$	$\frac{4}{5}$
A_0	$-\frac{8}{9}$	$\frac{64}{81}$	0	$\frac{24}{25}$
A_1	$\frac{10}{9}$	$\frac{40}{81}$	$\frac{40}{49}$	$\frac{2}{5}$
A_2	$\frac{1}{9}$	$\frac{25}{81}$	$\frac{9}{49}$	$\frac{9}{25}$

TABLE 3. Elements of $Q_{2,1}$ quadrature formulas

	$Q_{3,1}^{[1]}$	$Q_{3,1}^{[2]}$	$Q_{3,1}^{[3]}$	$Q_{3,1}^{[4]}$
α^2	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{19}{30}$	$\frac{2}{3}$
λ^2	$\frac{5}{2}$	1	$\frac{10}{11}$	$\frac{5}{6}$
A_0	$-\frac{56}{9}$	$-\frac{32}{81}$	0	$\frac{8}{25}$
A_1	$\frac{20}{9}$	$\frac{80}{81}$	$\frac{320}{361}$	$\frac{4}{5}$
A_2	$\frac{1}{9}$	$\frac{25}{81}$	$\frac{121}{361}$	$\frac{9}{25}$

TABLE 4. Elements of $Q_{3,1}$ quadrature formulas

	$Q_{3,2}^{[1]}$	$Q_{3,2}^{[2]}$	$Q_{3,2}^{[3]}$	$Q_{3,2}^{[4]}$
α^2	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{10+\sqrt{5}}{15}$	$\frac{2}{3}$
λ^2	$\frac{7}{4}$	1	$\sqrt{5}-2$	$\frac{1}{2}$
A_0	$\frac{58}{45}$	$\frac{128}{81}$	0	$\frac{8}{5}$
A_1	$\frac{49}{90}$	$\frac{40}{81}$	$\frac{8\left(21-4\sqrt{5}\right)}{361}$	$\frac{2}{5}$
A_2	$\frac{1}{45}$	$\frac{5}{81}$	$\frac{109+48\sqrt{5}}{361}$	$\frac{1}{5}$

TABLE 5. Elements of $Q_{3,2}$ quadrature formulas

$f(x,y)$	$\frac{1}{(3+x+y)^2}$	e^{xy}	$\sqrt{2+x+y}$	$\frac{1}{\sqrt{3+x+y}}$
$I_2[f]$	$5.87787E-01$	$4.22889E+00$	$5.51634E+00$	$2.38405E+00$
$Q_{2,1}^{[1]}[f]$	$6.06351E-01$	$4.24137E+00$	$5.48365E+00$	$2.38611E+00$
$Err_{2,1}^{[1]}$	$1.86E-02$	$1.25E-02$	$3.27E-02$	$2.06E-03$
$Q_{2,1}^{[2]}[f]$	$5.86676E-01$	$4.22897E+00$	$5.51752E+00$	$2.38394E+00$
$Err_{2,1}^{[2]}$	$1.11E-03$	$8.05E-05$	$1.18E-03$	$1.08E-04$
$Q_{2,1}^{[3]}[f]$	$5.93612E-01$	$4.23365E+00$	$5.51298E+00$	$2.38477E+00$
$Err_{2,1}^{[3]}$	$5.83E-03$	$4.76E-03$	$3.36E-03$	$7.24E-04$
$Q_{2,1}^{[4]}[f]$	$5.85275E-01$	$4.22800E+00$	$5.51830E+00$	$2.38376E+00$
$Err_{2,1}^{[4]}$	$2.51E-03$	$8.92E-04$	$1.96E-03$	$2.87E-04$

TABLE 6. Numerical results ($n = 2$ and $k = 1$)

$f(x,y,z)$	$\frac{1}{(4+x+y+z)^3}$	e^{xyz}	$\sqrt{3+x+y+z}$	$\frac{1}{\sqrt{4+x+y+z}}$
$I_3[f]$	$2.06717E-01$	$8.15085E+00$	$1.36405E+01$	$4.10778E+00$
$Q_{3,2}^{[1]}[f]$	$2.70857E-01$	$8.48274E+00$	$1.35969E+01$	$4.11385E+00$
$Err_{3,2}^{[1]}$	$6.41E-02$	$3.32E-01$	$4.35E-02$	$6.07E-03$
$Q_{3,2}^{[2]}[f]$	$2.12208E-01$	$8.27150E+00$	$1.36385E+01$	$4.10871E+00$
$Err_{3,2}^{[2]}$	$5.49E-03$	$1.21E-01$	$1.96E-03$	$9.34E-04$
$Q_{3,2}^{[3]}[f]$	$2.10618E-01$	$8.25999E+00$	$1.36390E+01$	$4.10850E+00$
$Err_{3,2}^{[3]}$	$3.90E-03$	$1.09E-01$	$1.41E-03$	$7.20E-04$
$Q_{3,2}^{[4]}[f]$	$2.09377E-01$	$8.25046E+00$	$1.36395E+01$	$4.10833E+00$
$Err_{3,2}^{[4]}$	$2.66E-03$	$9.96E-02$	$9.78E-04$	$5.47E-04$

TABLE 7. Numerical results ($n = 3$ and $k = 1$)

$f(x,y,z)$	$\frac{1}{(4+x+y+z)^3}$	e^{xyz}	$\sqrt{3+x+y+z}$	$\frac{1}{\sqrt{4+x+y+z}}$
$I_3[f]$	$2.06717E{-}01$	$8.15085E{+}00$	$1.36405E{+}01$	$4.10778E{+}00$
$Q_{3,2}^{[1]}[f]$	$2.12259E{-}01$	$8.09655E{+}00$	$1.36344E{+}01$	$4.10788E{+}00$
$Err_{3,2}^{[1]}$	$5.54E{-}03$	$5.43E{-}02$	$6.07E{-}03$	$9.88E{-}05$
$Q_{3,2}^{[2]}[f]$	$2.00868E{-}01$	$8.05430E{+}00$	$1.36426E{+}01$	$4.10692E{+}00$
$Err_{3,2}^{[2]}$	$5.85E{-}03$	$9.65E{-}02$	$2.11E{-}03$	$8.60E{-}04$
$Q_{3,2}^{[3]}[f]$	$2.00429E{-}01$	$8.01713E{+}00$	$1.36427E{+}01$	$4.10692E{+}00$
$Err_{3,2}^{[3]}$	$6.29E{-}03$	$1.34E{-}01$	$2.22E{-}03$	$8.64E{-}04$
$Q_{3,2}^{[4]}[f]$	$1.99127E{-}01$	$8.02972E{+}00$	$1.36432E{+}01$	$4.10668E{+}00$
$Err_{3,2}^{[4]}$	$7.59E{-}03$	$1.21E{-}01$	$2.71E{-}03$	$1.10E{-}03$

TABLE 8. Numerical results ($n = 3$ and $k = 2$)