STUDIA UNIV. "BABEŞ–BOLYAI", INFORMATICA, Volume **XLIV**, Number 2, 1999

CONCURRENCY-DEGREES FOR PETRI NETS

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ABSTRACT. The goal of this paper is to extend some concepts (about concurrency-degrees) from the class of Place/Transition Petri nets (PTN) to the class of jumping Petri nets ($JPTN$). Also, we will present a simpler definition of concurrency-degree for PTN. Moreover, we will point out how we can compute these concurrency-degrees.

Keywords: distributed systems, concurrency, Petri nets, jumping Petri nets, concurrency-degrees.

1. Introduction

A Petri net is a mathematical model used for the specification and the analysis of parallel/distributed systems. It is very important to introduce a measure of concurrency for parallel/distributed systems. What is the meaning of the fact that in the system S_1 the concurrency is greater than in the system S_2 ? We will study the problem of concurrency for Petri nets, but, since the Petri nets are used as suitable models for real parallel/distributed systems, the results will be applicable also to these systems.

It is well-known that the behaviour of some distributed systems cannot be adequately modelled by classical Petri nets. Many extensions which increase the computational and expressive power of Petri nets have been thus introduced. One direction has led to various modifications of the firing rule of nets. One of these extension is that of jumping Petri net, introduced in [TiJ94].

The notion of concurrency-degree for Petri nets was first introduced in [TJD93]. In this paper we will give a simpler definition of concurrency-degree for Petri nets, and we will extend this notion for jumping Petri nets. Also, we will show how we can compute these concurrency-degrees.

The paper is organized as follows. Section 2 presents the basic terminology, notation and results concerning Petri nets and jumping Petri nets. In section 3, and respectively 4, we present the definition of concurrency-degree for Petri nets, respectively for jumping Petri nets, and we show how we can compute these

1991 *CR Categories and Descriptors.* D.2.2 [**Software Engineering**]: Tools and Techniques – *Petri nets*; F.1.2 [**Computation by Abstract Devices**]: Modes of Computation – *Parallelism and concurrency* .

concurrency-degrees. Finally, in section 5 we conclude this paper and formulate some open problems.

2. Preliminaries

In this section we will establish the basic terminology, notation, and results concerning Petri nets in order to give the reader the necessary prerequisites for the understanding of this paper (for details the reader is referred to [BeF86], [JuT99], [Rei85], [Rei87]). Mainly, we will follow [JuT99], [TiJ94], [TiM97].

2.1. **Petri nets.** A *Place/Transition net*, shortly *P/T-net* or *net*, (finite, with infinite capacities), abbreviated *PTN*, is a 4-tuple $\Sigma = (S, T; F, W)$, where S and T are two finite non-empty sets (of *places* and *transitions*, resp.), $S \cap T = \emptyset$, $F \subseteq (S \times T) \cup (T \times S)$ is the *flow relation* and $W : (S \times T) \cup (T \times S) \to \mathbf{N}$ is the *weight function* of Σ verifying $W(x, y) = 0$ iff $(x, y) \notin F$.

A *marking* of a *PTN* Σ is a function $M : S \to \mathbf{N}$; it will be sometimes identified with a vector $M \in \mathbf{N}^{|S|}$. The operations and relations on vectors are componentwise defined. $\mathbf{N}^S$ denotes the set of all markings of Σ.

A *marked PTN*, abbreviated *mPTN*, is a pair $\gamma = (\Sigma, M_0)$, where Σ is a *PTN* and M_0, called the *initial marking* of γ, is a marking of Σ.

In the sequel we often use the term "Petri net" (PN) or "net" whenever we refer to a *PTN* (*mPTN*) γ and it is not necessary to specify its type (i.e. marked or unmarked).

Let γ be a net, $t \in T$ and $w \in T^*$. The functions $t^-, t^+ : S \to \mathbf{N}$ şi $\Delta t, \Delta w : S \to \mathbf{Z}$ are defined by $t^-(s) = W(s, t)$, $t^+(s) = W(t, s)$, $\Delta t(s) = t^+(s) - t^-(s)$ and

$$\Delta w(s) = \begin{cases} 0, & \text{if } w = \lambda, \\ \sum_{i=1}^{n} \Delta t_i(s), & \text{if } w = t_1 t_2 \ldots t_n (n \geq 1), \end{cases} \quad \text{for all } s \in S.$$

The sequential behaviour of a net γ is given by so-called *firing rule*, which consist of

- the *enabling rule*: a transition t is *enabled* at a marking M in γ (or t is *fireable* from M), abbreviated $M[t\rangle_\gamma$, iff $t^- \leq M$;
- the *computing rule*: if $M[t\rangle_\gamma$, then t may *occur* yielding a new marking M', abbreviated
  $M[t\rangle_\gamma M'$, defined by $M' = M + \Delta t$.

The notation "$[.\rangle_\gamma$" will be simplified to "$[.\rangle$" whenever γ is understood from context.

In fact, for any transition t of γ we have a binary relation on $\mathbf{N}^S$, denoted by $[t\rangle_\gamma$ and given by: $M[t\rangle_\gamma M'$ iff $t^- \leq M$ and $M' = M + \Delta t$. If $t_1, t_2, \ldots, t_n$, $n \geq 1$, are transitions of γ, $[t_1 t_2 \ldots t_n\rangle_\gamma$ will denote the classical product of the relations $[t_1\rangle_\gamma, \ldots, [t_n\rangle_\gamma$, i.e. $[t_1 t_2 \ldots t_n\rangle_\gamma = [t_1\rangle_\gamma \circ \ldots \circ [t_n\rangle_\gamma$. Moreover, we consider the relation $[\lambda\rangle_\gamma$ given by $[\lambda\rangle_\gamma = \{(M, M) | M \in \mathbf{N}^S\}$.

Let γ be a marked Petri net, $M \in \mathbf{N}^S$ and M_0 its initial marking. The word $w \in T^*$ is called a *transition sequence* from M in γ if there exists a marking M' of γ such that $M[w\rangle_\gamma M'$. Moreover, the marking M' is called *reachable* from M in γ. We denote by $TS(\gamma, M) = \{w \in T^* | M[w\rangle_\gamma\}$ the set of all transition sequence from M in γ, and by $RS(\gamma, M) = [M\rangle_\gamma = \{M' \in \mathbf{N}^S | \exists w \in TS(\gamma, M) : M[w\rangle_\gamma M'\}$ the set of all reachable markings from M in γ.

In the case $M = M_0$, the set $TS(\gamma, M_0)$ is abbreviated by $TS(\gamma)$, and the set $RS(\gamma, M_0)$ is abbreviated by $RS(\gamma)$ (or $[M_0\rangle_\gamma$) and it is called *the set of all reachable markings* of γ.

The marking M is *coverable* in γ if there exists a marking $M' \in [M_0\rangle_\gamma$ such that $M \leq M'$.

Let γ be a P/T-net, and $T' \subseteq T$ a set of transitions, which is called *step*. The step-type concurrent behaviour of the net γ is given by so-called *step firing rule*, which consist of

- the *step enabling rule*: a step T' is *concurrently enabled* at a marking M in γ (or T' is *fireable* from M), abbreviated $M[T'\rangle_\gamma$, iff $\sum_{t \in T'} t^- \leq M$;
- the *step computing rule*: if $M[T'\rangle_\gamma$, then T' may *occur* yielding a new marking M', abbreviated $M[T'\rangle_\gamma M'$, defined by $M' = M + \sum_{t \in T'} \Delta t$.

2.2. **Jumping Petri nets.** Jumping Petri nets ([TiJ94], [TiM97]) are an extension of classical nets, which allows them to do "spontaneous jumps" from a marking to another one (this is similar to λ-moves in automata theory).

A *jumping P/T-net*, abbreviated *JPTN*, is a pair $\gamma = (\Sigma, R)$, where Σ is a *PTN* and R, called the *set of (spontaneous) jumps* of γ, is a binary relation on the set of markings of Σ (i.e. $R \subseteq \mathbf{N}^S \times \mathbf{N}^S$). In what follows the set R of jumps of any *JPTN* will be assumed *recursive*, that is for any couple of markings (M, M') we can effectively decide whether or not $(M, M') \in R$.

A *marked jumping net*, abbreviated *mJPTN*, is defined similarly as an *mPTN*, by changing "Σ" into "Σ, R".

Let $\gamma = (\Sigma, R)$ be a *JPTN*. The pairs $(M, M') \in R$ are referred to as *jumps* of γ. If γ has finitely many jumps (i.e. R is finite) then we say that γ is a *finite jumping net*, abbreviated *FJPTN*.

We shall use the term "*jumping net*" (*JN*) ("*finite jumping net*" (*FJN*), resp.) to denoted a *JPTN* or a *mJPTN* (a *FJPTN* or a *mFJPTN*, resp.) whenever it is not necessary to specify its type (i.e. marked or unmarked).

Pictorially, a jumping Petri net will be represented as a classical net and, moreover, the relation R will be separately listed.

The behaviour of a jumping net γ is given by the *j-firing rule*, which consist of

- the *j-enabling rule*: a transition t is *j-enabled* at a marking M (in γ), abbreviated $M[t\rangle_{\gamma,j}$, iff there exists a marking M_1 such that $MR^*M_1[t\rangle_\Sigma$ (Σ being the underlying net of γ and R^* the reflexive and transitive closure of R);

- the *j-computing rule*: if $M[t\rangle_{\gamma,j}$, then the marking M' is *j-produced* by occurring t at M, abbreviated $M[t\rangle_{\gamma,j}M'$, iff there exists two markings M_1, M_2 such that $MR^*M_1[t\rangle_\Sigma M_2R^*M'$.

The notation "$[.\rangle_{\gamma,j}$" will be simplified to "$[.\rangle_j$" whenever γ is understood from the context.

The notions of *transition j-sequence* and *j-reachable marking* are defined similarly as for Petri nets (the relation $[\lambda\rangle_{\gamma,j}$ is defined by $[\lambda\rangle_{\gamma,j} = \{(M,M')|M,M' \in \mathbf{N}^S, MR^*M'\}$).

The *set of all j-reachable markings* of a marked jumping net γ is denoted by $RS(\gamma)$ or by $[M_0\rangle_{\gamma,j}$ (M_0 being the initial marking of γ).

The marking M is *coverable* in γ if there exists a marking $M' \in [M_0\rangle_{\gamma,j}$ such that $M \leq M'$.

Some jumps of a marked jumping net may be never used. Thus we say that a marked jumping net $\gamma = (\Sigma, R, M_0)$ is *R-reduced* ([TiJ94]) if for any jump $(M,M') \in R$ of γ we have $M \neq M'$ and $M \in [M_0\rangle_{\gamma,j}$.

3. Concurrency-degrees for P/T nets

The notion of concurrency-degree for Petri nets was first introduced in [TJD93] (that definition can be found also in [JuT99]). Here we will give a simpler definition of this notion.

First, let us recall the definition of a (maximal) step:

Definition 3.1. *Let $\gamma = (S,T;F,W)$ be a Petri net and M an arbitrary marking of γ.*
i) $T' \subseteq T$ is called a set of transitions concurrently enabled at M *(or, briefly, a* step at M*) if $\sum_{t\in T'} t^- \leq M$;*
ii) $T' \subseteq T$ is called a maximal set of transitions concurrently enabled at M *(or, briefly, a* maximal step at M*) if T' is a step at M and, for each $t \in T - T'$, $T' \cup \{t\}$ is not a step at M.*

Notation 3.1. *Let $\gamma = (S,T;F,W)$ be a Petri net and M an arbitrary marking of γ.*
1) We denote by $T(M)$ the set of all transitions enabled at the marking M, i.e.

$$T(M) = \{t \in T|\ t^- \leq M\}\ ;$$

2) We denote by $CT(M)$ the set of all subsets of transitions concurrently enabled at M, i.e.

$$CT(M) = \{T' \subseteq T|\sum_{t\in T'} t^- \leq M\}\ ;$$

3) We denote by $MCT(M)$ the set of all maximal subsets of transitions concurrently enabled at the marking M, i.e.

$$MCT(M) = \{T' \subseteq T|T' \text{ is a maximal step at } M\}\ .$$

Generally speaking, there exist more maximal subsets of transitions concurrently enabled at a marking M. Moreover, the maximality of sets w.r.t. concurrency does not imply the maximality of sets w.r.t. cardinality of sets (i.e. we can have two maximal steps at M, T_1 and T_2, with $|T_1| < |T_2|$).

Example 3.1. *For the marked Petri net γ represented in figure 1, it is easy to see that the subsets $T' = \{t_1, t_2, t_3, t_4\}$ and $T'' = \{t_1, t_2, t_5\}$ are maximal steps at the initial marking of γ, and, moreover, these are the only ones, i.e. $MCT(M_0) = \{T', T''\}$.*

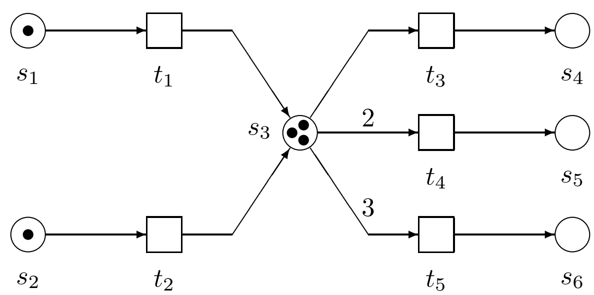


FIGURE 1. the net from example 3.1

Definition 3.2. *Let γ be a Petri net and M an arbitrary marking of γ. The* concurrency-degree at the marking M *of the net γ is defined by:*

$$d(\gamma, M) = \max\{\ |T'| \mid T' \in MCT(M)\}\ .$$

Definition 3.3. *Let $\gamma = (\Sigma, M_0)$ be a marked P/T-net.*
i) The inferior concurrency-degree *of the net γ is defined by:*

$$d^-(\gamma) = \min\{\ d(\gamma, M) \mid M \in [M_0\rangle_\gamma\}\ ;$$

ii) The superior concurrency-degree *of the net γ is defined by:*

$$d^+(\gamma) = \max\{\ d(\gamma, M) \mid M \in [M_0\rangle_\gamma\}\ .$$

Remark 3.1. *Directly from definitions we have*
1) $0 \le d^-(\gamma) \le d^+(\gamma) \le |T|$;
2) The inferior concurrency-degree of the net γ, $d^-(\gamma)$, represents the minimum number of transitions concurrently enabled at any reachable marking of γ, and has the property that there exists at least one reachable marking $M \in [M_0\rangle_\gamma$ such that there exists $d^-(\gamma)$ transitions concurrently enabled at M;
3) The superior concurrency-degree of the net γ, $d^+(\gamma)$, represents the maximum number of transitions concurrently enabled at any reachable marking of γ, and has the property that there exists at least one reachable marking $M \in [M_0\rangle_\gamma$ such that there exists $d^+(\gamma)$ transitions concurrently enabled at M.

Definition 3.4. *Let* $\gamma = (\Sigma, M_0)$ *be a marked Petri net. If* $d^-(\gamma) = d^+(\gamma)$*, then we denote this number with* $d(\gamma)$*, i.e.* $d(\gamma) = d^-(\gamma) = d^+(\gamma)$*, and we called it the* concurrency-degree *of* γ*.*

Example 3.2. *For the marked Petri net* γ *represented in figure 2, it is easy to see that transition* t_1 *is fireable from any reachable marking, i.e.* $t_1 \in T(M)$*, for all* $M \in [M_0\rangle_\gamma$*, which means that the inferior concurrency-degree is at least one:* $d^-(\gamma) \geq 1$*. Let* M_1 *be the marking produced by the occurence of* t_2 *at the initial marking, i.e.* $M_0[t_2\rangle_\gamma M_1$ *;* $M_0 = (2, 1, 0, 0, 0)$ *and* $M_1 = (1, 0, 0, 1, 0)$*. Since* t_1 *is the only transition fireable at* M_1*, i.e.* $T(M_1) = \{t_1\}$*, we have that* $d(\gamma, M_1) = 1$*, and therefore the inferior concurrency-degree is* $d^-(\gamma) = 1$*. Moreover, it is easy to see that the transition* t_2 *can occur at most one time in any transition sequence starting from the initial marking, and that the transition* t_3 *can occur also at most one time, and only after the occurence of* t_2*. This means that the set* $T = \{t_1, t_2, t_3\}$ *cannot be a step at any reachable marking, thus we have* $d^+(\gamma) < 3$*. Since the subset* $T' = \{t_1, t_2\}$ *is the only maximal step at* M_0*, i.e.* $MCT(M_0) = \{T'\}$*, we have that* $d(\gamma, M_0) = 2$*. Thus, the superior concurrency-degree is* $d^+(\gamma) = 2$*.*

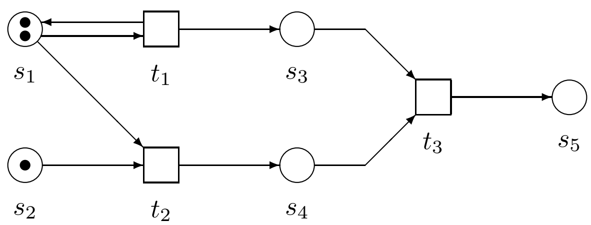


FIGURE 2. the net from example 3.2

In the sequel, we will show how we can compute the concurrency-degrees of a Petri net.

First of all, we will present the algorithm for computing the concurrency-degree at any marking of a Petri net.

Let γ be a Petri net, let M be an arbitrary marking, and let $|T| = n$. Obviously, $d(\gamma, M) \leq n$. The algorithm is the following:

Theorem 3.1. *The concurrency-degree at a marking,* $d(\gamma, M)$*, is computable for any PTN* γ *and for any marking* M*.*

Proof. It is easy to prove that the above algorithm is finite (i.e. it always stops) and it computes exactly the concurrency-degree at the marking M of γ. □

The complexity of the algorithm is $\mathcal{O}(2^{|T|} \cdot |S|)$.

```
procedure concurrency_degree_at_a_marking (γ: PTN, M: marking);
begin
for i := n downto 0 do   // where n = |T|
begin
// consider all the subsets of T with i elements
for each T' ⊆ T such that |T'| = i  do
begin
M' := Σ_{t∈T'} t^-; // M' is the smallest marking at which T' is concurrently enabled
if M' ≤ M then goto STOP;
end;
end;
STOP:  d(γ, M) := i ;
return d(γ, M);
end.
```

Now, we will present the algorithm for computing the superior concurrency-degree of a marked Petri net. Let $\gamma = (\Sigma, M_0)$ be a $mPTN$, and let $|T| = n$. Obviously, $d^+(\gamma) \leq n$. The algorithm is the following:

```
procedure superior_concurrency_degree (γ: mPTN);
begin
for i := n downto 0 do   // where n = |T|
begin
// consider all the subsets of T with i elements
for each T' ⊆ T such that |T'| = i  do
begin
M' := Σ_{t∈T'} t^-; // M' is the smallest marking at which T' is concurrently enabled
if  is_coverable(γ,M')  then goto STOP;
end;
end;
STOP:  d^+(γ) := i ;
return d^+(γ);
end.
```

```
boolean function is_coverable (γ: mPTN, M: marking);
begin
Let MCG(γ) be the minimal coverability graph of γ;
if  (there exists at least one node M' in MCG(γ) such that M ≤ M')
then return true  else return false;
end.
```

Theorem 3.2. *The superior concurrency-degree $d^+(\gamma)$ is computable for any mPTN γ.*

Proof. Since the coverability problem is decidable for *mPTN* ([KaM69]) (the function is_coverable solves this problem by using the minimal coverability graph, [Fin93]), it is easy to prove that the above algorithm is finite (i.e. it always stops) and it computes exactly the superior concurrency-degree of the net γ. $\square$

The complexity of the algorithm is $\mathcal{O}(2^{|T|}) \cdot \mathcal{O}(CP)$, where $\mathcal{O}(CP)$ is the complexity of the coverability problem solved using the minimal coverability graph (for details, see [Fin93]).

Now, we will show how we can compute the inferior concurrency-degree of a marked Petri net.

Let $\gamma = (\Sigma, M_0)$ be a marked Petri net. First, let us remark that if the reachability set $[M_0\rangle_\gamma$ is finite (this problem is decidable, [KaM69]), then we can compute the inferior concurrency-degree of γ by using directly the definition: $d^-(\gamma) = \min\{\, d(\gamma, M) \mid M \in [M_0\rangle_\gamma\}$, because the minimum is computed on a finite set and $d(\gamma, M)$ is computable, for each marking M (theorem 3.1).

Now, let us consider that the reachability set $[M_0\rangle_\gamma$ is infinite. Then, there exists a finite subset $\mathcal{M} \subseteq [M_0\rangle_\gamma$ such that

$$(*) \quad \forall M \in [M_0\rangle_\gamma, \exists M' \in \mathcal{M} \text{ such that } M' \leq M.$$

Indeed, we can consider $\mathcal{M}$ as being the set of minimal reachable markings of γ, i.e.

$$\mathcal{M} = \{M \in [M_0\rangle_\gamma \mid \forall M' \in [M_0\rangle_\gamma - \{M\} \ : \ M' \not\leq M\}.$$

Then, we have the result:

Proposition 3.1. *The following equality holds:*

$$\min\{d(\gamma, M) | M \in [M_0\rangle_\gamma\} = \min\{d(\gamma, M) | M \in \mathcal{M}\}.$$

Proof. This equality follows easily from $(*)$ and from the fact that the concurrency-degree at a marking is a monotone increasing function, i.e. $M_1 \leq M_2 \Rightarrow d(\gamma, M_1) \leq d(\gamma, M_2)$. $\square$

The following result about the usual quasi-ordering (i.e. the quasi-ordering on components) on $\mathbf{N}^k$ is well-known:

Lemma 3.1. (Dickson's lemma, [Dic13])
The usual quasi-ordering on $\mathbf{N}^k$ is a well quasi-ordering (i.e. from every infinite sequence of elements from $\mathbf{N}^k$, we can extract an infinite increasing sequence).

Proceeding from Dickson's lemma, it follows that any subset of $\mathbf{N}^k$ contains only finitely many incomparable vectors. Since, by its definition, the elements of $\mathcal{M}$ are incomparable, it follows that $\mathcal{M}$ is a finite set. Thus, since the set $\mathcal{M}$ of minimal reachable markings of the P/T-net γ is computable, from proposition 3.1 it follows that:

Theorem 3.3. *The inferior concurrency-degree $d^-(\gamma)$ is computable for any mPTN γ.*

4. Concurrency-degrees for jumping Petri nets

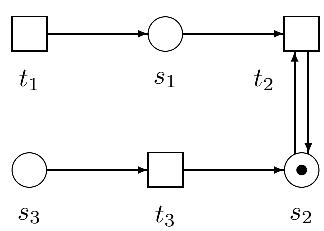
A jumping Petri net is a classical net Σ equipped with a (recursive) binary relation R on the markings of Σ. The meaning of a pair $(M, M') \in R$ is that the net Σ may "spontaneously jump" from M to M' (this is similar to λ-moves in automata theory).

We presented the definitions regarding jumping Petri nets in section 2. Now, we will present first an example of a jumping Petri net.

Example 4.1. *Let us consider a system consisting of a producer and a consumer, and a buffer with unlimited capacity, used for storing the products produced by the producer and consumed by the consumer. Moreover, we assume that the producer may take a break in any moment, and the consumer may take a break only when the buffer is empty (i.e., only when there are no products to consume).*

Such a system cannot be modelled by a classical Petri net ([JuT99]*). A modelling by an inhibitor Petri net was presented in* [JuT99]. *Here we will present a modelling of this system by a jumping Petri net.*

Let $\gamma = (\Sigma, R, M_0)$ be the marked jumping Petri net represented in figure 3. The place s_1 models the unlimited buffer, the transition t_1 models the producing of a product by the producer, and the transition t_2 models the consuming of a product by the consumer. The place s_2 models the active state of the consumer, and the place s_3 models the inactive state of him (i.e., the consumer is in a break). The fact that the consumer may take a break only when the buffer is empty, is modelled by the jump of this net, from the initial marking $M_0 = (0,1,0)$ to the marking $M_0' = (0,0,1)$, and the resuming of its activity by the transition t_3.



$$R = \{((0,1,0), (0,0,1))\}$$

Figure 3. the jumping net from example 4.1

Now, we will extend the notion of concurrency-degrees from P/T-nets to jumping Petri nets.

Definition 4.1. *Let $\gamma = (\Sigma, R)$ be a jumping Petri net, Σ being the underlying P/T-net of γ, and let M be an arbitrary marking of γ. The* concurrency-degree at the marking M *of the net γ is defined by:*

$$d(\gamma, M) = \max\{\ d(\Sigma, M') \mid MR^*M'\ \}\ .$$

Definition 4.2. *Let $\gamma = (\Sigma, R, M_0)$ be a marked jumping Petri net.*
i) The inferior concurrency-degree *of the net γ is defined by:*

$$d^-(\gamma) = \min\{\ d(\gamma, M) \mid M \in [M_0\rangle_{\gamma,j}\}\ ;$$

ii) The superior concurrency-degree *of the net γ is defined by:*

$$d^+(\gamma) = \max\{\ d(\gamma, M) \mid M \in [M_0\rangle_{\gamma,j}\}\ .$$

Moreover, the remarks about concurrency-degrees of P/T-nets (remark 3.1) hold for jumping Petri nets as well.

Definition 4.3. *Let $\gamma = (\Sigma, R, M_0)$ be a marked jumping Petri net. If $d^-(\gamma) = d^+(\gamma)$, then we denote this number with $d(\gamma)$, i.e. $d(\gamma) = d^-(\gamma) = d^+(\gamma)$, and we called it the* concurrency-degree *of the net γ.*

Example 4.2. *Let us recall the mFJPTN γ from example 4.1. We denote by M_n, M'_n the following markings: $M_n = (n, 1, 0)$, $M'_n = (n, 0, 1)$, for all $n \geq 0$. Thus, the set of jumps is $R = \{(M_0, M'_0)\}$, and it is easy to see that transition t_1 is fireable from any j-reachable marking, transition t_2 is fireable from all markings M_n, $n \geq 1$, and transition t_3 is fireable from all markings M'_n, $n \geq 0$. Therefore, the j-reachability set is $[M_0\rangle_{\gamma,j} = \{M_n | n \geq 0\} \cup \{M'_n | n \geq 0\}$, and the j-reachability graph of γ, $\mathcal{RG}(\gamma)$ (defined in* [ViJ99]*), is shown in figure 4.*

More precisely, $M_0RM'_0$ is the only jump in γ, and $T(M_0) = \{t_1\}$, with $M_0[t_1\rangle_\Sigma M_1$, which means that $d(\Sigma, M_0) = 1$. For all $n \geq 1$ we have that $T(M_n) = \{t_1, t_2\}$, with $M_n[t_1\rangle_\Sigma M_{n+1}$ and $M_n[t_2\rangle_\Sigma M_{n-1}$; moreover, $M_n[\{t_1, t_2\}\rangle_\Sigma M_n$ and $MCT(M_n) = \{\{t_1, t_2\}\}$. Therefore, $d(\Sigma, M_n) = 2$, $\forall n \geq 1$. Also, for all $n \geq 0$ we have that $T(M'_n) = \{t_1, t_3\}$, with $M'_n[t_1\rangle_\Sigma M'_{n+1}$ and $M'_n[t_3\rangle_\Sigma M_n$; moreover, $M'_n[\{t_1, t_3\}\rangle_\Sigma M_{n+1}$ and $MCT(M'_n) = \{\{t_1, t_3\}\}$. So, $d(\Sigma, M'_n) = 2$, $\forall n \geq 0$.

Now, let us compute the concurrency-degrees of the jumping net γ. Since $M_0RM'_0$ is the only jump in γ, we have that $d(\gamma, M_0) = max\{d(\Sigma, M_0), d(\Sigma, M'_0)\} = 2$, $d(\gamma, M_n) = d(\Sigma, M_n) = 2$, $\forall n \geq 1$, and $d(\gamma, M'_n) = d(\Sigma, M'_n) = 2$, $\forall n \geq 0$. Therefore, $d(\gamma) = d^-(\gamma) = d^+(\gamma) = 2$, so the concurrency-degree of γ is 2.

Let us notice that the inferior, respectively superior concurrency-degree of the underlying P/T-net of γ is $d^-(\Sigma) = 1$, resp. $d^+(\Sigma) = 2$; moreover, $d(\Sigma)$ is undefined, and the reachability set is $[M_0\rangle_\Sigma = \{M_n | n \geq 0\}$.

In the sequel, we will show how we can compute the concurrency-degrees of a jumping net.

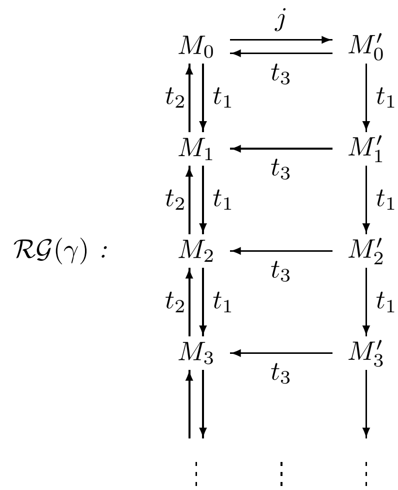


FIGURE 4. the j-reachability graph of the net γ

First of all, let us notice that the concurrency-degree at any marking of a *JPTN* γ can be computed if γ has the property: $\{M' \mid MR^*M'\}$ is finite, for each marking M (this follows easily from the definition 4.1 and the theorem 3.1, because we have to compute a maximum on a finite set). Let us observe that any finite jumping net has this property. Therefore, we have the result:

Theorem 4.1. *The concurrency-degree at a marking, $d(\gamma, M)$, is computable for any FJPTN γ and for any marking M.*

Now, let us notice that the algorithm for computing the superior concurrency-degree of a marked Petri net from section 3 works also for marked finite jumping Petri nets, because the coverability problem is decidable for *mFJPTN* ([TiJ94]) (the function is_coverable from section 3 solves the coverability problem for *mFJPTN* by using the minimal coverability graph, [ViJ99]). As a consequence, we have the result:

Theorem 4.2. *The superior concurrency-degree $d^+(\gamma)$ is computable for any mFJPTN γ.*

Now, we will show how we can compute the inferior concurrency-degree of a marked finite jumping Petri net.

Let $\gamma = (\Sigma, R, M_0)$ be a *mFJPTN*. First, let us remark that if the reachability set $[M_0\rangle_{\gamma,j}$ is finite (this problem is decidable, [TiJ94]), then we can compute the inferior concurrency-degree of γ by using directly the definition: $d^-(\gamma) = \min\{\ d(\gamma, M) \mid M \in [M_0\rangle_{\gamma,j}\}$, because the minimum is computed on a finite set and $d(\gamma, M)$ is computable, for each marking M (theorem 4.1).

Now, let us consider that the reachability set $[M_0\rangle_{\gamma,j}$ is infinite. Then, there exists a finite subset $\mathcal{M} \subseteq [M_0\rangle_{\gamma,j}$ such that

$$(*) \quad \forall M \in [M_0\rangle_{\gamma,j}, \exists M' \in \mathcal{M} \text{ such that } M' \leq M.$$

Indeed, we can consider $\mathcal{M}$ as being the set of minimal reachable markings of γ, i.e.

$$\mathcal{M} = \{M \in [M_0\rangle_{\gamma,j} | \forall M' \in [M_0\rangle_{\gamma,j} - \{M\} : M' \not\leq M\}.$$

Then, we have the result:

Proposition 4.1. *The following equality holds:*

$$\min\{d(\gamma, M) | M \in [M_0\rangle_{\gamma,j}\} = \min\{d(\gamma, M) | M \in \mathcal{M}\}.$$

Proof. This equality follows easily from $(*)$ and from the fact that the concurrency-degree at a marking is a monotone increasing function, i.e. $M_1 \leq M_2 \Rightarrow d(\gamma, M_1) \leq d(\gamma, M_2)$. □

Proceeding from Dickson's lemma (lemma 3.1), it follows that any subset of $\mathbf{N}^k$ contains only finitely many incomparable vectors. Since, by its definition, the elements of $\mathcal{M}$ are incomparable, it follows that $\mathcal{M}$ is a finite set.

Let us show how the set $\mathcal{M}$ can be constructed. Let $\gamma = (\Sigma, R, M_0)$ be a *mFJPTN*, with $R \neq \emptyset$, i.e.

$$R = \{ (M'_i, M''_i) | 1 \leq i \leq n\}, \quad n \geq 1,$$

such that $M'_i \in [M_0\rangle_{\gamma,j}$ (this can be done, see [TiJ94]). As in [TiJ94], we associate to γ the following *mPTN*s:

$$\gamma_0 = (\Sigma, M_0) \text{ and } \gamma_i = (\Sigma, M''_i), \text{ for each } 1 \leq i \leq n,$$

and then, let $\mathcal{M}_i$ be the set of minimal reachable markings of γ_i, for each $1 \leq i \leq n$. These sets are finite (it follows from Dickson's lemma) and we have that:

$$\mathcal{M} = \{M \in \mathcal{M}' \mid \forall M' \in \mathcal{M}' - \{M\} \ : \ M' \not\leq M\},$$

where $\mathcal{M}' = \cup\{\mathcal{M}_i | 1 \leq i \leq n\}$. Thus, since the sets $\mathcal{M}_i$, $1 \leq i \leq n$, are computable, the set $\mathcal{M}$ is also computable, and from proposition 4.1 it follows that:

Theorem 4.3. *The inferior concurrency-degree $d^-(\gamma)$ is computable for any mFJPTN γ.*

5. Conclusions

In this paper we have extended some concepts (mainly, concurrency-degrees) from the class of Place/Transition Petri nets (*PTN*) to the class of jumping Petri nets (*JPTN*). Also, we have presented a simpler definition of concurrency-degree for *PTN* and we have shown how we can compute concurrency-degrees.

Many problems remain to be studied, for example:

- finding an efficient algorithm for computing the set $\mathcal{M}$ for Petri nets;

- finding better algorithms for computing the concurrency-degrees for Petri nets;
- extending the computability results regarding concurrency-degrees for *FJPTN* for the larger class of jumping Petri nets.

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