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THE LINEAR NORMED SPACE $\overline{S(G)}$ OF THE NETWORKS ATTACHED TO A GRAPH G

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ABSTRACT. In this paper we define the linear normed space $\overline{S(G)}$ of all networks attached to a graph G. We also define and study several convergence in $\overline{S(G)}$. We adopt the definition of network as metric space introduced by P. M. Dearing and R. L. Francis (1974) and the functional representation of network introduced by E. Iacob in (1997).

1. Preliminary notions and results

The definition of network as metric space was introduced in [2] and was used in [4], [6], [5], etc.

We consider an undirected, connected graph $G = (W, A)$, without loops or multiple edges. To each vertex $w_i \in W$ we associate a point v_i from the euclidean space $\mathbf{R}^q, q \geq 2$. This yields a finite subset V of $\mathbf{R}^q$, called the vertex set of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset \mathbf{R}^q$ called edge of the network. We assume that any two edges have no interior common points. We denote by $E = \{e_1, ..., e_m\}$, $e_k = [v_{i_k}, v_{j_k}]$, $k = 1, 2, ..., m$, the set of all edges. We define the network $N = (V, E)$ by

$$N = \{x \in \mathbf{R}^q \mid \exists\, (w_i, w_j) \in A \text{ so that } x \in [v_i, v_j]\}. \tag{1}$$

We suppose that for each edge $e_k \in E$ there exist a continuous mapping $T_{e_k} : [0,1] \to [v_{i_k}, v_{j_k}]$ so that

$$T_{e_k}(0) = v_{i_k}, T_{e_k}(1) = v_{j_k} \text{ and } T_{e_k}([0,1]) = [v_{i_k}, v_{j_k}].$$

As in [5] we functionally represent the network $N = (V, E)$ by

$$\begin{gathered} \varphi_N : [0,1] \to \mathbf{R}^{q \times m}, \\ \varphi_N(t) = (T_{e_1}(t), \ldots, T_{e_m}(t)). \end{gathered} \tag{2}$$

For every function T_{e_k}, $k = 1, 2, ..., m$, we denote the corresponding scalar components with $T_{e_k 1}, \ldots, T_{e_k q}$, hence we have

$$T_{e_k}(t) = (T_{e_k 1}(t), \ldots, T_{e_k q}(t)).$$

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Remark 1.1. *We can associate to each vertex $w_i \in W$ of the graph G an infinity points v_i from the Euclidean space $\mathbf{R}^q$ and we also can associate to each edge (w_i, w_j) of the graph G an infinity of rectifiable arcs $[v_i, v_j]$ in $\mathbf{R}^q$. So to the graph G we can associate an infinity of networks denoted by $S(G)$.*

Definition 1.1. [5] *Two networks $N_1 = (V_1, E_1)$, $N_2 = (V_2, E_2)$ are called homologous if there is a one-to-one application $h : V_1 \to V_2$ so that if $[v_1, v_2] \in E_1$ then $[h(v_1), h(v_2)] \in E_2$. In this case the edges $[v_1, v_2] \in E_1$ and $[h(v_1), h(v_2)] \in E_2$ are called homologous.*

Remark 1.2. *The set $S(G)$ is the set of all the homologous networks with the network N defined in (1).*

In [5] are introduced the following operations on $S(G)$. We consider two networks

$$N_1 = (V_1, E_1), N_2 = (V_2, E_2), N_1, N_2 \in S(G),$$

functionally represented by

$$\begin{aligned} \varphi_{N_i} &: [0,1] \to \mathbf{R}^{q \times m}, \\ \varphi_{N_i}(t) &= \left(T^i_{e_1}(t), \ldots, T^i_{e_m}(t)\right), \ i = 1, 2 \end{aligned} \tag{3}$$

so that the edges $T^1_{e_k}, T^2_{e_k}$ are homologous $\forall k = 1, 2, ..., m$. Further on we suppose that the corresponding edges $T^1_{e_k}, T^2_{e_k}$ are homologous for every two networks $N_1, N_2 \in S(G)$ functionally represented as above.

Definition 1.2. [5]*1. The sum of the networks $N_1, N_2 \in S(G)$ is a network $N_1 + N_2 \in S(G)$, functionally represented by*

$$\begin{aligned} \varphi_{N_1+N_2} &: [0,1] \to \mathbf{R}^{q \times m}, \\ \varphi_{N_1+N_2}(t) &= \varphi_{N_1}(t) + \varphi_{N_2}(t), t \in [0,1]. \end{aligned}$$

2. The multiplication with a scalar $\lambda \in \mathbf{R}^*$ of the network $N_1 \in S(G)$, is a network $\lambda N_1 \in S(G)$, functionally represented by

$$\begin{aligned} \varphi_{\lambda N_1} &: [0,1] \to \mathbf{R}^{q \times m}, \\ \varphi_{\lambda N_1}(t) &= \lambda \varphi_{N_1}(t), t \in [0,1]. \end{aligned}$$

We define now the null network associated to the graph G. To each vertex $w_i \in W$ we associate the point $\theta = (0, 0, \ldots, 0) \in \mathbf{R}^q$ and to each edge $(w_{i_k}, w_{j_k}) \in A$ we associate the function

$$T_{e_k} : [0,1] \to \theta, T_{e_k}(t) = \theta, \forall t \in [0,1].$$

Definition 1.3. *The application*

$$\varphi_0 : [0,1] \to \mathbf{R}^{q \times m}, \varphi_0(t) = (T_{e_1}(t), \ldots, T_{e_m}(t)),$$

with

$$T_{e_k}(t) = \theta, \forall t \in [0,1], \forall k = 1, 2, ..., m$$

is called the null network and is denoted by $\mathbf{0}$.

We denote $\overline{S(G)} = S(G) \cup \mathbf{0}$.

Finally we denote

$$C_{\mathbf{R}^q}[0,1] = \{f : [0,1] \to \mathbf{R}^q \mid f \text{ is continuous on } [0,1]\}$$

and we recall the following norma in $C_{\mathbf{R}^q}[0,1]$.

Definition 1.4. [1] *The application*

$$\|\cdot\| : C_{\mathbf{R}^q}[0,1] \to \mathbf{R}, \|f\| = \max_{t\in[0,1]} \|f(t)\|,$$

where $\|f(t)\|$ *is the euclidean norma of the element* $f(t) \in \mathbf{R}^q$, *is a norma in* $C_{\mathbf{R}^q}[0,1]$.

2. Properties of the operations with networks

Lemma 2.1. *The operation of addition of two networks from* $\overline{S(G)}$ *defined above is associative and commutative, that is*

$$(N_1 + N_2) + N_3 = N_1 + (N_2 + N_3)$$

and

$$N_1 + N_2 = N_2 + N_1, \forall N_1, N_2, N_3 \in \overline{S(G)}.$$

Proof. We consider the networks $N_i(V_i, E_i), i = 1, 2, 3,$ from $\overline{S(G)}$, functionally represented by

$$\begin{aligned} \varphi_{N_i} &: [0,1] \to \mathbf{R}^{q\times m}, \\ \varphi_{N_i}(t) &= \left(T^i_{e_1}(t), \ldots, T^i_{e_m}(t)\right), i = 1, 2, 3, \end{aligned}$$

so that the edges $T^1_{e_k}, T^2_{e_k}, T^3_{e_k}$ are homologous $\forall k = 1, 2, ..., m$.We have:

$$\varphi_{(N_1+N_2)+N_3}(t) = \varphi_{N_1+N_2}(t) + \varphi_{N_3}(t) = \varphi_{N_1}(t) + \varphi_{N_2}(t) + \varphi_{N_3}(t)$$

and

$$\varphi_{N_1+(N_2+N_3)}(t) = \varphi_{N_1}(t) + \varphi_{N_2+N_3}(t) == \varphi_{N_1}(t) + \varphi_{N_2}(t) + \varphi_{N_3}(t),$$

$\forall t \in [0,1]$.Consequently

$$\varphi_{(N_1+N_2)+N_3}(t) = \varphi_{N_1+(N_2+N_3)}(t), \forall t \in [0,1],$$

hence $(N_1 + N_2) + N_3 = N_1 + (N_2 + N_3)$.

The other affirmation can be proved in a similar way. □

Lemma 2.2. *For every network* $N = (V, E) \in \overline{S(G)}$, *functionally represented by* (2), *we have* $N + \mathbf{0} = \mathbf{0} + \mathbf{N} = \mathbf{N}$.

Proof. Indeed,

$$\varphi_{N+\mathbf{0}}(t) = \varphi_N(t) + \varphi_{\mathbf{0}}(t) = \varphi_N(t)$$

and

$$\varphi_{\mathbf{0}+\mathbf{N}}(t) = \varphi_{\mathbf{0}}(t) + \varphi_N(t) = \varphi_N(t), \forall t \in [0,1],$$

hence

$$N + \mathbf{0} = \mathbf{0} + \mathbf{N} = \mathbf{N}, \forall N \in \overline{S(G)}.$$

□

We consider now a network $N = (V, E) \in \overline{S(G)}$, functionally represented by (2), and we denote with $-N$ the network from $\overline{S(G)}$, functionally represented by

$$\varphi_{-N} : [0,1] \to \mathbf{R}^{q \times m}, \varphi_{-N}(t) = \left(-T_{e_1}(t), \ldots, -T_{e_m}(t)\right), \forall t \in [0,1].$$

The network $-N$ is called the opposite network of N.

Lemma 2.3. *For every network $N = (V, E) \in \overline{S(G)}$, functionally represented by (2), we have*

$$N + (-N) = -N + N = \mathbf{0}.$$

Proof. For every $N = (V, E) \in \overline{S(G)}$,

$$\begin{aligned} \varphi_{N+(-\mathbf{N})}(t) &= \varphi_{-N+N}(t) = \varphi_N(t) + \varphi_{-\mathbf{N}}(t) = \\ &= \varphi_N(t) - \varphi_{\mathbf{N}}(t) = \theta, \forall t \in [0,1]. \end{aligned}$$

□

From Lemma 2.1, Lemma 2.2 and Lemma 2.3 we obtain the next theorem.

Theorem 2.1. *The set $\overline{S(G)}$ is a commutative group related with the first operation defined in Definition 1.2.*

Lemma 2.4. *For every $\lambda \in \mathbf{R}$ and for every $N_1, N_2 \in \overline{S(G)}$ we have*

$$\lambda (N_1 + N_2) = \lambda N_1 + \lambda N_2.$$

Proof. Indeed,

$$\varphi_{\lambda(N_1+N_2)}(t) = \lambda \varphi_{N_1+N_2}(t) = \lambda \varphi_{N_1}(t) + \lambda \varphi_{N_2}(t), \forall t \in [0,1].$$

□

Lemma 2.5. *For every $\lambda, \mu \in \mathbf{R}$ and for every $N \in \overline{S(G)}$ we have $(\lambda + \mu) N = \lambda N + \mu N$.*

Proof. It is clear that

$$\varphi_{(\lambda+\mu)N}(t) = (\lambda + \mu)\, \varphi_N(t) = \lambda \varphi_N(t) + \mu \varphi_N(t), \forall t \in [0,1].$$

□

Lemma 2.6. *For every $\lambda, \mu \in \mathbf{R}$ and for every $N \in \overline{S(G)}$ we have $\lambda(\mu N) = (\lambda\mu)N$.*

Proof. Indeed,

$$\varphi_{\lambda(\mu N)}(t) = \lambda \varphi_{\mu N}(t) = (\lambda\mu)\varphi_N(t), \forall t \in [0,1].$$

□

Lemma 2.7. *If 0 and 1 are the null element and the unit element from $\mathbf{R}$, then $0 \cdot N = \mathbf{0}$ and $1 \cdot N = N$, $\forall N \in \overline{S(G)}$.*

Proof. We have

$$\varphi_{\,0\cdot N}(t) = 0 \cdot \varphi_N(t) = \varphi_{\mathbf{0}}(t), \forall t \in [0,1]$$

and

$$\varphi_{1\cdot N}(t) = 1 \cdot \varphi_N(t) = \varphi_N(t), \forall t \in [0,1].$$

□

From the Theorem 2.1 and Lemma 2.4, Lemma 2.5, Lemma 2.6, Lemma 2.7 we obtain:

Theorem 2.2. *The set $\overline{S(G)}$ endowed with the operations from the Definition 1.2 is a real linear space.*

We will endowed now the set $\overline{S(G)}$ with a normed space structure.

We consider a network $N \in \overline{S(G)}$, functionally represented by (2). Because for every edge $e_k = [v_{i_k}, v_{j_k}]$, $k = 1, 2, ..., m$ of the network N correspond a continuous application $T_{e_k} : [0,1] \to [v_{i_k}, v_{j_k}]$ we define the norma of the edge e_k as the norma of the application T_{e_k} from the Definition 1.4.

Definition 2.1. *The norma of the edge e_k, $k = 1, 2, ..., m$ of the network $N \in \overline{S(G)}$ is:*

$$\|e_k\| = \max_{t\in[0,1]} \|T_{e_k}(t)\|. \tag{4}$$

Proposition 2.1. *The application*

$$\|\cdot\| : \overline{S(G)} \to \mathbf{R},\ \|N\| = \max_{k=1,2,...,m} \|e_k\| = \max_{k=1,2,...,m} \max_{t\in[0,1]} \|T_{e_k}(t)\|, \tag{5}$$

$\forall N \in \overline{S(G)}$, is a norma in the linear space $\overline{S(G)}$, that is it satisfy the following properties:

(1) $\|N\| \geq 0$, $\forall N \in \overline{S(G)}$ and $\|N\| = 0$ if and only if $N = \mathbf{0}$;
(2) $\|\lambda N\| = |\lambda|\ \|N\|$, $\forall \lambda \in \mathbf{R}$, $\forall N \in \overline{S(G)}$;
(3) $\|N_1 + N_2\| \leq \|N_1\| + \|N_2\|$, $\forall N_1, N_2 \in \overline{S(G)}$.

Proof.

(1) From the relation (5) it is obviously that $\|N\| \geq 0$, $\forall N \in \overline{S(G)}$. We also have

$$\begin{aligned}\|N\| &= 0 \Leftrightarrow \max_{k=1,2,...,m} \max_{t\in[0,1]} \|T_{e_k}(t)\| = 0 \Leftrightarrow T_{e_k}(t) = \\ &= \theta, \forall t \in [0,1], \forall k = 1, 2, ..., m \Leftrightarrow N = \mathbf{0}.\end{aligned}$$

(2) For $\forall \lambda \in \mathbf{R}$ and $\forall N \in \overline{S(G)}$,

$$\begin{aligned}\|\lambda N\| &= \max_{k=1,2,...,m} \max_{t\in[0,1]} \|\lambda T_{e_k}(t)\| = \\ &= |\lambda| \max_{k=1,2,...,m} \max_{t\in[0,1]} \|T_{e_k}(t)\| = |\lambda|\ \|N\|.\end{aligned}$$

(3) We consider the networks $N_1, N_2 \in \overline{S(G)}$ functionally represented as in (3). For $\forall k = 1, 2, ..., m$ we have

$$\left\|T^1_{e_k}(t) + T^2_{e_k}(t)\right\| \leq \left\|T^1_{e_k}(t)\right\| + \left\|T^2_{e_k}(t)\right\|, \forall t \in [0,1].$$

This implies

$$\begin{aligned}\max_{t\in[0,1]} \left\|T^1_{e_k}(t) + T^2_{e_k}(t)\right\| &\leq \max_{t\in[0,1]} \left\{\left\|T^1_{e_k}(t)\right\| + \left\|T^2_{e_k}(t)\right\|\right\} \leq \\ &\leq \max_{t\in[0,1]} \left\|T^1_{e_k}(t)\right\| + \max_{t\in[0,1]} \left\|T^2_{e_k}(t)\right\|\end{aligned}$$

and

$$\begin{aligned}&\max_{k=1,2,...,m} \left\{\max_{t\in[0,1]} \left\|T^1_{e_k}(t) + T^2_{e_k}(t)\right\|\right\} \\ \leq\ &\max_{k=1,2,...,m} \left\{\max_{t\in[0,1]} \left\|T^1_{e_k}(t)\right\| + \max_{t\in[0,1]} \left\|T^2_{e_k}(t)\right\|\right\} \leq \\ \leq\ &\max_{k=1,2,...,m}\max_{t\in[0,1]} \left\|T^1_{e_k}(t)\right\| + \max_{k=1,2,...,m}\max_{t\in[0,1]} \left\|T^2_{e_k}(t)\right\|,\end{aligned}$$

that is $\|N_1 + N_2\| \leq \|N_1\| + \|N_2\|$.

□

Further on we can organize the linear normed space $\overline{S(G)}$ as a metric space if we define the distance between two networks $N_1, N_2 \in \overline{S(G)}$, functionally represented as in (3), by

$$\rho(N_1, N_2) = \|N_1 - N_2\| = \max_{k=1,2,...,m}\max_{t\in[0,1]} \left\|T^1_{e_k}(t) - T^2_{e_k}(t)\right\|. \tag{6}$$

If we consider the metric space $\left(\overline{S(G)}, \rho\right)$ with the distance ρ defined in (6) then it is naturally to make a study of convergent sequences of elements from $\overline{S(G)}$ in the sense of the metric ρ. We will study in the next section several types of convergence in $\overline{S(G)}$.

3. Convergence in $\overline{S(G)}$

Several mathematicians were studying sequences of different mathematical objects (see [8], [1], [3], etc.). We will study now the sequences of networks from $\overline{S(G)}$.

We consider the networks

$$N_1 = (V^1, E^1), N_2 = (V^2, E^2), ..., N_n = (V^n, E^n), ... \tag{7}$$

of $\overline{S(G)}$ where $V^1, V^2, \ldots, V^n, \ldots$ are respectively the vertex sets and $E^1, E^2, ..., E^n, ...$ are respectively the edges sets of the networks $N_1, N_2, ..., N_n, ...$ We denote this sequence of networks by $(N_n)_{n\in\mathbb{N}^*}$ or simple by (N_n).

For any $n \geq 1$ we denote the edges of the networks N_n by $e_1^n, e_2^n, ..., e_m^n$, hence $E^n = \{e_1^n, e_2^n, ..., e_m^n\}$. We suppose that for every $k = 1, 2, ..., m$ the corresponding edges e_k^n are homologous for every $n \geq 1$.

Using the notation (2) we functionally represent the networks of the sequence (7) by:

$$\varphi_{N_n} : [0,1] \to \mathbf{R}^{q\times m}, n = 1, 2, ...$$
$$\varphi_{N_n}(t) = \left(T_{e_1}^n(t), T_{e_2}^n(t), ..., T_{e_m}^n(t)\right),$$

where the functions

$$T_{e_k}^n : [0,1] \to e_k^n = \left[v_{i_k}^n, v_{j_k}^n\right], \ \forall k = 1, 2, ..., m$$

and $\forall n \geq 1$ are continuous so that:

$$T_{e_k}^n(0) = v_{i_k}^n, T_{e_k}^n(1) = v_{j_k}^n \text{ and } T_{e_k}^n([0,1]) = \left[v_{i_k}^n, v_{j_k}^n\right].$$

Related to the sequence (7) we study different problems of convergence. Thus for example for a number $t \in [0,1]$, the sequence

$$T_{e_k}^1(t), T_{e_k}^2(t), ..., T_{e_k}^n(t), ...$$

can be convergent or not in $\mathbf{R}^q$ for some fixed $k \in \{1, 2, ..., m\}$ or even for every $k \in \{1, 2, ..., m\}$. Moreover these can be happen for every $t \in [0,1]$.

We consider a fixed integer number $k \in \{1, 2, ..., m\}$.

Definition 3.1. *The sequence (N_n) of $\overline{S(G)}$ is k-convergent on the point $t \in [0,1]$ if the sequence*

$$T_{e_k}^1(t), T_{e_k}^2(t), ..., T_{e_k}^n(t), ... \tag{8}$$

is convergent in $\mathbf{R}^q$. In this case we denote $\lim_{n\to\infty} T_{e_k}^n(t) = T_{e_k}(t)$.

Definition 3.2. *The sequence (N_n) of $\overline{S(G)}$ is convergent on the point $t \in [0,1]$ if the sequence (8) is convergent in $\mathbf{R}^q$, for all $\forall k \in \{1, 2, ..., m\}$. In this case*

$$\lim_{n\to\infty} T_{e_k}^n(t) = T_{e_k}(t), \forall k \in \{1, 2, ..., m\}.$$

Definition 3.3. *The sequence (N_n) of $\overline{S(G}$ is k-simple convergent if $\forall t \in [0,1]$, the sequence (8) is convergent. So we can define the limit function*

$$T_{e_k} : [0,1] \to \mathbf{R}^q$$
$$T_{e_k}(t) = \lim_{n\to\infty} T_{e_k}^n(t).$$

Definition 3.4. *The sequence (N_n) of $\overline{S(G)}$ is simple convergent if $\forall k \in \{1, 2, ..., m\}$ and $\forall t \in [0,1]$, the sequence (8) is convergent.*

In this case we obtain m functions

$$T_{e_k} : [0,1] \to \mathbf{R}^q, k = 1, 2, ..., m.$$

The fact that for some $k \in \{1, 2, ..., m\}$, and for $t \in [0,1]$ the sequence (8) is convergent in $\mathbf{R}^q$, its limit being $T_{e_k}(t)$, means: $\forall \varepsilon > 0, \exists r(\varepsilon, t)$ so that

$$\left\|T^n_{e_k}(t) - T_{e_k}(t)\right\| < \varepsilon, \forall n > r. \tag{9}$$

Therefore r depend by ε and t. We desire to praise those sequences for that $\forall \varepsilon > 0$ there exist a number r who depend only by ε so that the inequality (9) is satisfied $\forall n > r$, and for any $t \in [0,1]$.

Definition 3.5. *The sequence (N_n) of $\overline{S(G)}$ is k-uniformly convergent if for any number $\varepsilon > 0$, there exist a number $r(\varepsilon)$, which do not depend by t, so that the inequality (9) is satisfied $\forall n > r$, and for any $t \in [0,1]$.*

Remark 3.1. *In this case the function $T_{e_k} : [0,1] \to \mathbf{R}^q$ is continuous on $[0,1]$ so this function define a edge between the points $T_{e_k}(0)$ and $T_{e_k}(1)$.*

Proof. We consider $t_0 \in [0,1]$. We proof that the function T_{e_k} is continuous on t_0. If the sequence (N_n) is k-uniformly convergent then $\forall\ \varepsilon > 0, \exists r(\varepsilon)$, which do not depend by t, so that the inequality $\left\|T^n_{e_k}(t) - T_{e_k}(t)\right\| < \varepsilon/3$ is satisfied $\forall n > r$, and for any $t \in [0,1]$. We consider a fixed number $n > r$. The function $T^n_{e_k}$ is continuous on t_0, hence for the number ε considered above there exist a neigborhood V_{t_0} of t_0 so that for any $t \in V_{t_0} \cap [0,1]$, $\left\|T^n_{e_k}(t) - T^n_{e_k}(t_0)\right\| < \varepsilon/3$. We obtain

$$\begin{aligned} \|T_{e_k}(t) - T_{e_k}(t_0)\| \leq \left\|T_{e_k}(t) - T^n_{e_k}(t)\right\| + \left\|T^n_{e_k}(t) - T^n_{e_k}(t_0)\right\| + \\ + \left\|T^n_{e_k}(t_0) - T_{e_k}(t_0)\right\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

for every $t \in V_{t_0} \cap [0,1]$, hence the function T_{e_k} is continuous on t_0. The point t_0 was considered arbitrar, so we obtain that the function T_{e_k} is continuous on $[0,1]$. □

Definition 3.6. *The sequence (N_n) of $\overline{S(G)}$ is uniformly convergent if it is k-uniformly convergent for any $k \in \{1, 2, \cdots, m\}$.*

In this case, using the Remark 3.1 we obtain m continuous functions $T_{e_k} : [0,1] \to \mathbf{R}^q$, $k \in \{1, 2, \cdots, m\}$ and a new network $N \in \overline{S(G)}$.

We consider a uniformly convergence sequence (N_n) of $\overline{S(G)}$.

Theorem 3.1. *The application*

$$\begin{aligned} \varphi_N &: [0,1] \to \mathbf{R}^{q \times m} \\ \varphi_N(t) &= \ (T_{e_1}(t), T_{e_2}(t), ..., T_{e_m}(t)) \end{aligned}$$

where

$$T_{e_k}(t) = \lim_{n \to \infty} T^n_{e_k}(t), \forall k \in \{1, 2, ..., m\}, \forall t \in [0,1]$$

functionally represents a network from $\overline{S(G)}$ called the limit network of the sequence (N_n).

Proof. The functions $T_{e_k} : [0,1] \to \mathbf{R}^q$ are continuous on $[0,1]$, $\forall k \in \{1,2,\cdots,m\}$. We consider now two adjacent edges $T^n_{e_i}, T^n_{e_j}$, of the networks N_n, $\forall n \geq 1$, so that $T^n_{e_i}(1) = T^n_{e_j}(0)$. Consequently $\lim\limits_{n\to\infty} T^n_{e_i}(1) = \lim\limits_{n\to\infty} T^n_{e_j}(0)$. But

$$T_{e_i}(1) = \lim_{n\to\infty} T^n_{e_i}(1), T_{e_j}(0) = \lim_{n\to\infty} T^n_{e_j}(0)$$

hence $T_{e_i}(1) = T_{e_j}(0)$. This implies that the edges T_{e_i} and T_{e_j} are adjacent too. □

Theorem 3.2. *The necessary and sufficient condition for the sequence (N_n) of $\overline{S(G)}$ to be uniformly convergent is that for any $\varepsilon > 0$ there exist a number r independent by t so that $\forall n, p > r$,*

$$\left\|T^p_{e_k}(t) - T^n_{e_k}(t)\right\| < \varepsilon, \tag{10}$$

$\forall k = 1, 2, ..., m$ and $\forall t \in [0,1]$.

Proof. We suppose first that the sequence (N_n) is uniformly convergent and it have the limit the network N. We consider a real number $\varepsilon > 0$. There exist then a number r which do not depend by t so that

$$\left\|T^l_{e_k}(t) - T_{e_k}(t)\right\| < \varepsilon/2 \tag{11}$$

for every $l > r$, $k = 1, 2, ..., m$ and $t \in [0,1]$. We apply the inequality (11) for $l = n$ and $l = p$, where $n > r$ and $p > r$. We have: $\left\|T^n_{e_k}(t) - T_{e_k}(t)\right\| < \varepsilon/2$ and $\left\|T^p_{e_k}(t) - T_{e_k}(t)\right\| < \varepsilon/2$ for every $k = 1, 2, ..., m$ and $\forall t \in [0,1]$, consequently $\left\|T^p_{e_k}(t) - T^n_{e_k}(t)\right\| < \varepsilon$ for every $k = 1, 2, ..., m$ and $\forall t \in [0,1]$.

Conversely, we suppose that for every $\varepsilon > 0$, there exist a number r independent by t so that $\forall n, p > r$, $\left\|T^p_{e_k}(t) - T^n_{e_k}(t)\right\| < \varepsilon, \forall k = 1, 2, ..., m$ and $\forall t \in [0,1]$. For every $t \in [0,1]$ and $k = 1, 2, ..., m$ the sequence $T^1_{e_k}(t), T^2_{e_k}(t), ..., T^n_{e_k}(t), ...$ is then fundamental and consequently convergent in $\mathbf{R}^q$. Hence there exist the limit function $T_{e_k} : [0,1] \to \mathbf{R}^q$, $T_{e_k}(t) = \lim\limits_{n\to\infty} T^n_{e_k}(t), \forall k = 1, 2, ..., m$. We fix the number n in the inequality (10) and we obtain the following inequality when the number p tend to ∞:

$$\left\|T^n_{e_k}(t) - T_{e_k}(t)\right\| < \varepsilon$$

for every $t \in [0,1]$ and $k = 1, 2, ..., m$. Consequently the sequence (N_n) is uniformly convergent. □

Remark 3.2. *If the sequence (N_n) is uniformly convergent then it is also simple convergent. The reciprocal is not true as it result from the next example:*

Example 3.1. [3] *We consider the graph $G = (W, A)$, $W = \{w_1, w_2\}$ and $A = \{(w_1, w_2)\}$. To the vertex w_1 and w_2 of the graph G we associate the points $v_1 = (0,0)$ respective $v_2 = (1,1)$ in $\mathbf{R}^2$. We consider the sequence (N_n) of $\overline{S(G)}$ functionally represented by*

$$\varphi_{N_n} : [0,1] \to \mathbf{R}^2, n = 1, 2, ...$$
$$\varphi_{N_n}(t) = (T^n_{e_1}(t)),$$

where
$$T^n_{e_1} : [0,1] \to [v_1^n, v_2^n]$$
$$T^n_{e_1}(t) = (t, t^n).$$

This sequence of networks is simple convergent because $\forall t \in [0,1]$, *the sequence*
$$T_{e_1}(t), T^2_{e_1}(t), ..., T^n_{e_1}(t), ...$$

is convergent and we obtain the limit function
$$T_{e_1} : [0,1] \to \mathbf{R}^2$$
$$T_{e_1}(t) = \begin{cases} (t,0), & \text{if } 0 \le t < 1 \\ (t,1), & \text{if } t = 1 \end{cases}.$$

But this function is not continuous, so using the Remark 3.1 the sequence (N_n) *is not uniformly convergent.*

We remark that we can define through continuity a new network $N^* \in \overline{S(G)}$ with the vertex
$$v_1^* = (0,0), v_2^* = (1,0),$$
functionally represented by
$$\varphi_{N^*} : [0,1] \to \mathbf{R}^2$$
$$\varphi_{N^*}(t) = (T^*_{e_1^*}(t)),$$
where
$$T^*_{e_1^*} : [0,1] \to [v_1^*, v_2^*]$$
$$T^*_{e_1^*}(t) = (t,0).$$

Now we introduce the convergence in the sense of the metric ρ introduced in (6).

Definition 3.7. *The sequence* (N_n) *of* $\overline{S(G)}$ *is convergent and have limit the network* $N \in \overline{S(G)}$ *if*
$$\lim_{n\to\infty} \rho(N_n, N) = 0$$
where ρ *is the distance defined in (6).*

This means: for any $\varepsilon > 0$, there exist a number $r(\varepsilon)$ so that the inequality $n > r(\varepsilon)$ implies the inequality $\rho(N_n, N) < \varepsilon$,that is
$$\max_{k=1,2,...,m} \left\{ \max_{t\in[0,1]} \left\| T^n_{e_k}(t) - T_{e_k}(t) \right\| \right\} < \varepsilon.$$

That is for any $\varepsilon > 0$, there exist a number $r(\varepsilon)$ which do not depend by $t \in [0,1]$ and by $k = 1,2,...,m$ so that $\left\| T^n_{e_k}(t) - T_{e_k}(t) \right\| < \varepsilon$ for any $t \in [0,1]$, $k = 1,2,...,m$ and $n > r(\varepsilon)$.

So the convergence of the sequence (N_n) to the network N in the sense of Definition 3.7 is equivalent with the uniformly convergence of the sequence (N_n) to N.

Theorem 3.3. *If the sequence (N_n) of $\overline{S(G)}$ is convergent in the sense of Definition 3.7 then its limit is unique.*

Proof. The proof is obviously because $\left(\overline{S(G)}, \rho\right)$ is a metric space and in a metric space the limit of a convergent sequence in the metric sense is unique. □

Definition 3.8. *The sequence (N_n) of networks in $\overline{S(G)}$ is fundamental if for any $\varepsilon > 0$, there exist a number $r(\varepsilon)$ so that for any $n > r(\varepsilon)$ and $p > r(\varepsilon)$, the following inequality is satisfied*

$$\rho(N_p, N_n) < \varepsilon$$

that is

$$\max_{k=1,2,\ldots,m} \left\{ \max_{t\in[0,1]} \left\| T^p_{e_k}(t) - T^n_{e_k}(t) \right\| \right\} < \varepsilon.$$

Theorem 3.4. *If the sequence (N_n) of networks in $\overline{S(G)}$ is convergent in the sense of Definition 3.7 then it is fundamental.*

Proof. We suppose that the sequence (N_n) of networks from $\overline{S(G)}$ is convergent and have the limit the network N. Hence for every $\varepsilon > 0$, there exist a number $r(\varepsilon)$ so that $\rho(N_n, N) < \varepsilon/2$ for every $n > r(\varepsilon)$. We consider the index n and p so that $n > r(\varepsilon)$ and $p > r(\varepsilon)$. Consequently there are satisfied the following inequalities: $\rho(N_n, N) < \varepsilon/2$, $\rho(N_p, N) < \varepsilon/2$. Hence we have:

$$\rho(N_p, N_n) \le \rho(N_p, N) + \rho(N, N_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

These imply that for every number $\varepsilon > 0$, there exist a number $r(\varepsilon)$, so that $d(N_p, N_n) < \varepsilon$ for every $p > r(\varepsilon)$ and $n > r(\varepsilon)$, hence the sequence of networks (N_n) is fundamental. □

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