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# Circular Distance in Directed Networks

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**Abstract.** In this article we define the circular distance in strongly directed networks. Its basic properties are studied. We adopt the definition of network as metric space introduced by Dearing and Francis (1974) and the definition of directed networks introduced by Iacob (1994).

We start with an undirected, connected graph $G = (W, A)$, without loops or multiple edges. To each vertex $w_i \in W = \{w_1, ..., w_n\}$ we associate a point $v_i$ from an euclidean space $X$. Thus yields a finite subset $V = \{v_1, ..., v_n\}$ of $X$, called the vertex set of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset X$ called edge of the network. We assume that any two edges have no interior common points. Consider that $[v_i, v_j]$ has the positive length $l_{ij}$ and denote by $U$ the set of all edges. We define the network $N = (V, U)$ by

$$N = \{x \in X \mid \exists (w_i, w_j) \in A \text{ so that } x \in [v_i, v_j]\}$$

It is obvious that $N$ is a geometric image of $G$, witch follows naturally from an embedding of $G$ in $X$. Suppose that for each $[v_i, v_j] \in U$ there exist a continuous one-two-one mapping $\theta_{ij} : [v_i, v_j] \to [0, 1]$ with $\theta_{ij}(v_i) = 0, \theta_{ij} : (v_j) = 1$, and $\theta_{ij}([v_i, v_j]) = [0, 1]$.

Any connected and closed subset of an edge bounded by two points $x$ and $y$ of $[v_i, v_j]$ is called a closed subedge and is denoted by $[x, y]$. If one or both of $x, y$ miss we say that the subedge is open in $x$, or in $y$ or is open and we denote this by $(x, y]$or $[x, y)$ or $(x, y)$,respectively. Using $\theta_{ij}$ , it is possible to compute the length of $[x, y]$ as $l([x, y]) = |\theta_{ij}(x) - \theta_{ij}(y)| \cdot l_{ij}$. Particularly we have $l([v_i, v_j]) = l_{ij}$ , $l([v_i, x]) = \theta_{ij}(x) l_{ij}$ and $l([x, v_j]) = (1 - \theta_{ij}(x)) \cdot l_{ij}$. By analogy with graphs we introduce the notion:

The degree $g_N(v)$ of $v \in V$ in $N$ is the number of closed edges which contain $v$. The vertex $v$ with $g_N(v) = 1$ is called terminal vertex.

We consider the points $x, y \in N$ and the following edges and at most two subedges at the extremities

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$$[x, v_1], [v_1, v_2], ..., [v_{k-1}, v_k], [v_k, y], k \in N, k \leq n, \{v_1, ..., v_k\} \subseteq V$$

when the vertex $v_1, ..., v_k$ are not indispensable distinctly.

A path $L(x, y)$ linking the point $x$ and $y$ in $N$ is the union of all edges and subedges in the sequence considered. If $x = y$ and $k \geq 3$ the path is called cycle. The length of a path (cycle) is the sum of the lengths of all its component edges and subedges and we denote it by $l(L(x, y))$. If a path (cycle) contains only distinct vertices then we call it elementary.

A network is connected if for any points $x, y \in N$ there exist a path $L(x, y) \subset N$.

A connected network without cycles is called tree.

Let $L^*(x, y)$ be a shortest path between the points $x, y \in N$. This path is also called geodesic. We define a distance on $N$ as follows: $d(x, y) = l(L^*(x, y))$, for any $x, y \in N$. It is obvious that $(N, d)$ is a metric space.

**Definition 1.** *A vertex $v$ of a network is calledf point of articulation if $N \setminus \{v\}$ is not connected.*

The directed networks was introduced by E. Iacob in [Ia94]. Because we already presented undirected networks we will specify only the difference between that and directed networks. If in an undirected network $N = (V, U)$ we attach at all edges a sense the result will be a directed network and edges with the respectively sense we will called arcs.

We denote an arc between vertex $v_i, v_j \in V$ also with $[v_i, v_j]$ and the sense of arc will be from initial vertex $v_i$ to final vertex $v_j$. If $x, y \in [v_i, v_j]$ then the subarc $[x, y]$ is a connected subset of $[v_i, v_j]$ having the same sense with arc $[v_i, v_j]$. The notion length of arc and subarc is identical with the length of edge respectively subedge. So in the sequel we suppose that the arc $[v_i, v_j]$ have a positive length $l_{ij}$.

A directed path $D(x, y)$ from the point $x \in N$ to point $y \in N$ in a directed networks $N$ is a path passing once through a vertex in witch all arcs and subarcs have the same sense witch is the sense of directed path too.

So is the lots of points who belong to arcs and subarcs of sequence

$$[x, v_1], [v_1, v_2], ..., [v_{k-1}, v_k], [v_k, y], k \in N,$$
$$k \leq n, \{v_1, , , v_k\} \subseteq V, |\{v_1, ... v_k\}| = k$$

If $x = y$ the directed path $D(x, y)$ is called directed cycle.

We denote with $D^*(x, y)$ a shortest directed path between the points $x, y \in N$.

**Definition 2.** *A directed networks $N$ is called strongly connected if for all $x, y \in N$ exist a directed path from $x$ to $y$.*

We will endow the directed networks $N$ with a metric space similar to the structure introduced for directed graphs by B.Zelynka in [Zel97]..

We recall that in an undirected graph the distance between two vertices is usually defined as the length of the shortest path connecting these vertices. This distance is a metric on the vertex set of the graph. Analogously in a directed graph (usually the strong connectedness is supposed) the distance $d(x, y)$ from a vertex $x$ to a vertex $y$ is defined as the length of the shortest directed path from $x$ to $y$. In general $d(x, y)$ thus defined is not a metric, because it is not symmetric. In [Zel97] B. Zelinka define a certain distance in a directed graph which is a metric, namely circular distance.

Let $G$ be a strongly connected directed graph and let $x, y$ be two vertices of $G$.

**Definition 3.** [Zel97] *The circular distance $d^{o}(x, y)$ between the vertices $x, y$ in the graph $G$ is defined as*

$$d^{0}(x, y) = d(x, y) + d(y, x)$$

*were $d$ denotes the usual distance in directed graphs (see above).*

**Proposition 4.** [Zel97] *The circular distance $d^{0}(x, y)$ is a metric on the vertex set $V(G)$ of the graph $G$.*

We consider in what follows a strongly connected network $N$ and $x, y \in N$.

**Definition 5.** *The distance from $x$ to $y$, $d(x, y)$ is the length of the shortest directed path from $x$ to $y$*

(1)
$$d(x, y) = l(D^{*}(x, y))$$

As in directed graphs $d(x, y)$thus defined is not a metric, because it is not symmetric.

Analogous to circular distance in directed graphs we define the circular distance in directed networks, which is a metric.

**Definition 6.** *The circular distance $d^{0}(x, y)$ between the points $x, y$ in the strongly connected network $N$ is*

(2)
$$d^{0}(x, y) = d(x, y) + d(y, x)$$

*where $d$ is the distance defined in [1].*

In other words $d^{0}(x, y)$ is the length of the shortest directed path going from $x$ to $y$ and then back to $x$. .

Note that in the mentioned path, called circular path, vertex and arcs may repeat.

In the directed network in fig.1 such circular path for $x$ and $y$ contain all arcs of the directed network and the arc $e$ occurs twice in it.

The following proposition is evident.

**Proposition 7.** *The circular distance $d^{0}(x, y)$ is a metric on $N$.*

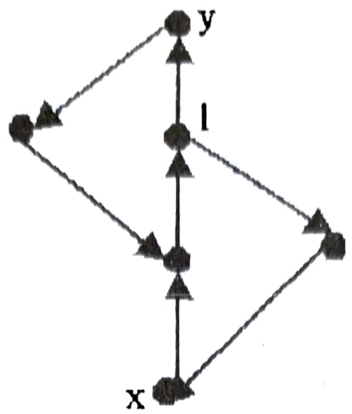


FIGURE 1

As in directed graphs, we define:

**Definition 8.** *The length of the shortest directed cycle in the directed network $N$ is called the directed girth of $N$ and is denoted by $g(N)$.*

**Proposition 9.** *If $x, y$ are two distinct points of a strongly connected network $N$, then*

$$d^0(x, y) \geq g(N)$$

*Proof.* Let $x, y \in N$. Then

$$d^0(x, y) = l(D^*(x, y)) + l(D^*(y, x)) \; \square$$

The union of directed path $D^*(x, y)$ and $D^*(y, x)$ must contain a directed cycle. The length of this cycle is greater than or equal to $g(N)$ and less than or equal to the sum of lengths of $D^*(x, y)$ and $D^*(y, x)$. This implies the assertion.

Analogously as for the usual distance, we may introduce the circular radius and circular diameter.

Let $x \in N = (V, E)$ and $A \subset N$. Then

$$e^0(x) = \max\{d^0(x, y) \mid y \in N\}$$

is called circular elongation of $x$ and

$$e^0(x, A) = \max\{d^0(x, y) \mid y \in A\}$$

is called circular elongation of $x$ relative to $A$.

**Definition 10.** *The circular radius $r^0$ of $N$ is*

$$r^0 = \min\{e^0(v_i, V) \mid v_i \in V\}$$

*The absolute circular radius $r_a^0$ of $N$ is*

$$r_a^0 = \min\{e^0(x, V) \mid x \in N\}$$

*The general circular radius $r_g^0$ of $N$ is*

$$r_g^0 = \min\{e^0(v_i) \mid v_i \in V\}$$

*The continuous circular radius $r_c^0$ of $N$ is*

$$r_c^0 = \min\left\{e^0(x) \mid x \in N\right\}$$

Any vertex $v_i \in V$ such that $e^0(v_i, V) = r^0$ is a circular center and the circular center set is thus

$$C^0 = \left\{v_i \in V \mid e^0(v_i, V) = r^0\right\}$$

Any point $x \in N$ such that $e^0(x, V) = r_a^0$ is circular absolute center and the circular absolute center set is

$$C_a^0 = \left\{x \in N \mid e^0(x, V) = r_a^0\right\}$$

Any vertex $v_i \in V$ such that $e^0(v_i) = r_g^0$ is a circular general center and the circular general center set is

$$C_g^0 = \left\{v_i \in V \mid e^0(v_i) = r_g^0\right\}$$

Any point $x \in N$ such that $e^0(x) = r_c^0$ is a circular continuous center and the circular continuous center set is

$$C_c^0 = \left\{x \in N \mid e^0(x) = r_c^0\right\}$$

Let us consider now the maximum circular distances between points. We denote the diameter of $N$ by:

$$\delta^0 = \max\left\{d^0(v_i, v_j) \mid v_i, v_j \in V\right\} = \max\left\{e^0(v_i, V) \mid v_i \in V\right\}$$

the absolute diameter by:

$$\delta_a^0 = \max\left\{d^0(x, v_i) \mid x \in N, v_i \in V\right\}$$

the continuous diameter by:

$$\delta_c^0 = \max\left\{d^0(x, y) \mid x, y \in N\right\}$$

**Proposition 11.** *For the circular radius $r^0$ and circular diameter $\delta^0$ of a strongly connected directed network $N$ the following inequality holds:*

$$r^0 \leq \delta^0 \leq 2r^0$$

*Proof.* The first inequality is true because the minimum of a set is less or equal than the maximum of a same set. For the second inequality we denote with $v_i$ an circular center of $N$, so

$$e^0(v_i, V) = r^0 = \min\left\{e^0(v_j, V) \mid v_j \in V\right\} \square$$

We denote with $v_k$ the vertex for who $e^0(v_k, V) = \delta^0$. In this case $\exists v_j \in V$ so that

$$\begin{aligned}\delta^0 = e^0(v_k, V) = d^0(v_k, v_j) \leq d^0(v_k, v_i) + d^0(v_i, v_j) \leq \\ \leq e^0(v_i, V) + e^0(v_i, V) = 2r^0\end{aligned}$$

so we have $\delta^0 \leq 2r^0$.

**Proposition 12.** *For the diameter, absolute diameter and continuous diameter of a strongly connected directed network N the following inequality holds:*

$$\delta^0 \leq \delta^0_a \leq \delta^0_c$$

*Proof.* By the definitions:

$$\begin{aligned} \delta^0 &= \max\left\{d^0\left(v_i, v_j\right) \mid v_i, v_j \in V\right\} \leq \delta^0_a = \\ &= \max\left\{d\left(x, v_i\right) \mid x \in N, v_i \in V\right\} \leq \\ &\leq \delta_c = \max\left\{d\left(x, y\right) \mid x, y \in N\right\} \end{aligned}$$

**Proposition 13.** *For the circular radius $r^0$ and for the absolute circular radius $r^0_a$ and the following inequality holds* .

$$r^0 \leq 2r^0_a$$

*Proof.* Let $c_a$ denote the circular absolute center, $v_k$ the circular center and $v_e$ a vertex such that $d^0\left(v_k, v_e\right) = r^0$. Both $d^0\left(v_k, c_a\right) \leq r^0_a$ and $d^0\left(v_e, c_a\right) \leq r^0_a$ by definition of $c_a$. By the triangle inequality

$$2r^0_a \geq d^0\left(v_k, c_a\right) + d^0\left(v_e, c_a\right) \geq d^0\left(v_k, v_e\right) = r^0$$

so we have $r^0 \leq 2r^0_a$. □

**Definition 14.** [La85] *A network N is a cactus if no two cycles have more than one vertex in common.*

Note that trees and cycles are special cases thereof.

**Definition 15.** *A directed cactus is any directed network which is obtained from a cactus by orienting its edge.*

The following proposition is easy to prove.

**Proposition 16.** *For any two distinct vertices $u, v$ of $N$ there exist a unique directed path from $u$ to $v$ in $N$ if and only if $N$ is a strongly connected directed cactus.*

Thus we see that in certain sense strongly connected directed cacti are analogues of trees in the case of directed networks. That form a particular case of the concept of unipathic directed networks defined as follows, analogously to unipathic digraphs defined in[Har65].

**Definition 17.** *A unipathic directed network is a directed network N in which for any two points $u, v \in N$ there exist at most one directed path from $u$ to $v$.*

Further on we prove:

**Theorem 18.** *If $x, y$ are two distinct vertices of a directed cactus network $N$, then $d^0\left(x, y\right)$ is equal to the sum of lengths of all cycles in $N$ which have common edges with the path $D\left(x, y\right)$.*

*Proof.* The proof is analogously with that done for Zelinka in[Zel97]for directed cactus graphs. □

We will proceed by induction according to the number $k$ of cycles which contain edges of $D(x, y)$. If $k = 1$, then $x$ and $y$ are in the same cycle $B$ and this cycle is the (edge-disjoint) union of $D(x, y)$ and $D(y, x)$, therefore $d^0(x, y)$is equal to the length of cycle $B$.

Now let $k \geq 2$ and suppose that for $k - 1$ the assertion is true. Let the first edge of $D(x, y)$ be in the cycle $B_1$and let $a$ be the terminal vertex of the last edge of $D(x, y)$ being in $B_1$. Then $a$ is an articulation between $B_1$ and another cycle $B_2$ which contains the edge of $D(x, y)$ outgoing from $a$. The path $D(a, y)$ is part of $D(x, y)$ and there are $k - 1$ cycles containing edges of $D(a, y)$, namely all those containing edges of $D(x, y)$ except $B_1$. By the induction hypotesis $d^0(a, y)$ is the sum of lengths of these cycles. Not only $D(x, y)$, but also $D(y, x)$ goes through $a$ and therefore $d^0(x, y) = d^0(x, a) + d^0(a, y)$, which is the sum of lengths of all cycles which contains edges of $D(x, y)$.

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