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On Some Extended Fuzzy Measures Applicable to Stochastic Automata

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Abstract. This paper studies the possibility to predict with a certain imprecision the external behaviour of a stochastic automaton where we don't have complete information concerning the output function. We express the lack of information by defining an output function which asssigns to each current internal state of the automaton a set of subsets A of output symbols. To each subset A we assign a probability measure $p(A)$. The probability measure means that we know that one symbol a of the corresponding subset A will be the output symbol (with the probability $p(A)$) but we don't know anything about how that symbol a is selected from the subset A of output symbols. Under these circumstances we define a confidence level which gives us some information about the chances of each symbol to be (at a ceratin moment) the output symbol.

1. Preliminaries

1.1. **Stochastic Automata.** Let X be a finite nonempty set called *alphabet*. The members of the set X are *symbols* or *letters*. By *sequence* (or *word* over X) we denote any finite string of symbols of X. To designate sequences we will use letters like u, v, w. By $l(w)$ we denote the number of symbols from X which compose the sequence w and we call $l(w)$ the *length* of w. With Λ we denote the empty word (or empty sequence) for which $l(\Lambda) = 0$. As known, concatenation, as binary operation, is defined for finite sequences according to the following: let $u = x_1 \ldots x_n$ and $v = y_1 \ldots y_m$ (where $x_i, y_j \in X$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$), then $uv := x_1 \ldots x_n y_1 \ldots y_m$. The result of concatenation is also a sequence over X. $X* := \{u | u \text{ is a sequence over } x\} \cup \{\Lambda\}$ denotes the free monoid over X under concatenation with identity .

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Definition 1.1. *A* ***deterministic Mealy finite automaton*** *is a system of the form* $A = \langle X, Y, S, \delta, \nu\rangle$ *where:*

1. X, Y *and* S *are finite nonempty sets designating the input alphabet, the output alphabet and the set of internal states of the automaton respectively;*
2. $\delta : X \times S \to S$ *is an application which establishes the new internal state, depending on the input symbol received by the automaton in the current state;*
3. $\nu : X \times S \to Y$ *is an application which establishes the output symbol as the response of the automaton to the input symbol received by the automaton in the current state.*

The applications δ and ν describe the internal and the external behaviour of the automaton, respectively. If the behaviour of an automaton follows probabilistic laws, we obtain:

Definition 1.2. *A* ***generalized stochastic automaton*** *is a system of the form* $A = \langle X, Y, S, \{p(s', y|s, x)\}\rangle$ *where:*

1. X, Y *and* S *are finite nonempty sets designating the input alphabet, the output alphabet and the set of internal states of the automaton, respectively;*
2. $\{p(s', y|s, x)\} = \{p(s', y|s, x) | x \in X, y \in Y, s, s' \in S\}$ *is the set of transition probabilities.* $p(s', y|s, x)$ *represents the conditional transition probability of the automaton from state* s *to state* s' *under the input* x *and assuming that the output symbol is* y.

The conditional transition probabilities satisfy $0 \le p(s', y|s, x) \le 1$ and

$$\sum_{s' \in S} \sum_{y \in Y} p(s', y|s, x) = 1$$

for all $(s, x) \in S \times X$. A generalized stochastic automaton is supposed to be a sequential and syncron machine. Its behaviour is described in matricial form by $P(y|x) := (p_{ij}(y|x))$, for $i = 1, \dots, n$, $j = 1, \dots, m$ and for all $x \in X$, $y \in Y$. Here $p_{ij}(y|x) := p(s_j, y|s_i, x)$ if we can order the internal states of the automaton. The sequential behaviour of the automaton is expressed by $P(vy|ux) = P(v|u)P(y|x)$ for all $x \in X$, $u \in X*$, $y \in Y$, $v \in Y*$.

Let the stochastic vector π_0 describe the fact that the automaton starts from one state with a certain probability. π_0 denotes the *initial state partition*. An automaton for which the initial state partition is defined, is called initial automaton. In the case of an *initial automaton* $\pi(y|x) = \pi_0 P(y|x)$.

Definition 1.3. *A* **stochastic Mealy automaton** *is a generalized stochastic automaton where for all* $s, s' \in S$ *and all* $x \in X$, $y \in Y$ *the following holds for the transition probabilities of the automaton:*

(1)
$$p(s', y|s, x) = p(s'|s, x)p(y|s, x)$$

If we use the notation $p(y|s, x, s') = p(s', y|s, x)/p(s'|s, x)$ where $p(s'|s, x) \neq 0$, it is obvious that in the case of a stochastic Mealy automaton we obtain $p(y|s, x, s') = p(y|s, x)$.

1.2. Fuzzy Measures.

Definition 1.4. *A fuzzy measure is determined by a function* $g : \mathcal{P}(X) \rightarrow [0, 1]$ *which assigns to each crisp subset of a universal set* X *a real number in* $[0, 1]$.

A fuzzy measure verifies the well-known axioms:

G1: $g(\emptyset) = 0$ and $g(X) = 1$;
G2: if $A \subseteq B$, than $g(A) \leq g(B)$ for all $A, B \in \mathcal{P}(X)$
G3: if X is not a finite set, then for each sequence $(A_i \in \mathcal{P}(X) | i \in N)$ of subsets of X for which $A_1 \subseteq A2 \subseteq \ldots$ or $A_1 \supseteq A2 \supseteq \ldots$, the following holds: $\lim_{i \to \infty} g(A_i) = g(\lim_{i \to \infty} A_i)$.

Definition 1.5. *A belief measure is a function* $Bel : \mathcal{P}(X) \rightarrow [0, 1]$ *that satisfies the axioms G1-G3 and additionaly, the following:*

$$Bel(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \sum_i Bel(A_i) - \sum_{i<j} Bel(A_i \cap A_j) + \cdots (-1)^{n+1} Bel(A_1 \cap \cdots \cap A_n)$$

for all $n \in N$ *and all families of subsets of* X.

Definition 1.6. *A plausibility measure is a function* $Pls : \mathcal{P}(X) \rightarrow [0, 1]$ *that satisfies the axioms G1-G3 and additionaly, the following:*

$$Pls(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \sum_i Pls(A_i) - \sum_{i<j} Pls(A_i \cup Aj) + \cdots (-1)^{n+1} Pls(A_1 \cup \cdots \cup An)$$

for all $n \in N$ *and all families of subsets of* X.

It is easy to show that, for all subsets A of X, the following hold:

(2) $$\mathrm{Bel}(A) = 1 - \mathrm{Pls}(XA) \quad \mathrm{Pls}(A) = 1 - \mathrm{Bel}(XA)$$

By XA we denote the complement of A.

It is possible to express the belief measure as well as the corresponding plausibility measure using the application $m : \mathcal{P}(X) \rightarrow [0, 1]$, called basic *probability assignment*. The application m satisfies $m(\emptyset) = 0$ and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$ and m is not a fuzzy measure.

Given a basic probability assignment m, one can determine uniquely the belief as well as the corresponding plausibility measure, for all subsets A of X, according to the relations:

(3) $$\mathrm{Bel}(A) = \sum_{B \subseteq A} m(B) \quad \text{and} \quad \mathrm{Pls}(A) = \sum_{B \cap A \neq \emptyset} m(B)$$

It is easy to show that $\mathrm{Pls}(A) \geq \mathrm{Bel}(A)$.

A is called focal element of m if $m(A) > 0$. The pair $(\mathcal{F}, m)$ represents a body of evidence if $\mathcal{F}$ is a set of focal elements and m stands for the corresponding basic probability assignment.

2. Stochastic Automaton with Imprecise and Uncertain Output Function

Definition 2.1. *A stochastic automaton with imprecise output function is a system of the form* $A = \langle X, Y, S, \{p(s'|s,x)\}, \mu, \pi_0\rangle$ *where:*

- **(i):** X, Y *and* S *are finite nonempty sets designating the input alphabet, the output alphabet and the set of internal states;*
- **(ii):** $\{p(s'|s,x)\}$ *represents the set of values of the conditional transition probabilities of the automaton;*
- **(iii):** $\pi_0 : S \rightarrow [0,1]$ *denotes the initial state partition of the automaton;*
- **(iv):** $\mu : S \rightarrow (\mathcal{P}(Y) \times [0,1])$ *is the output function which designates the possible output symbols corresponding to each current internal state of the automaton.*

μ denotes an application of the form: $\mu(s_j) = \{<A_{1j}, p_{1j}>, <A_{2j}, p_{2j}>, \ldots, <A_j^r, p_j^r>\}$ for each state $s_j \in S$ and for the subsets A_j^i of Y for which p_j^i has strict positive values.

The following relations hold:

$$0 \le p_j^i \le 1, \text{ (for } i = 1, \ldots, r) \text{ and } \sum_{i=1}^{r} p_j^i = 1 \tag{4}$$

for every j, index of any internal state $s_j \in S$.

Note that r in the above relations depends on the state s_j.

The output function μ maps an internal state of the automaton into a set of pairs where the first element is a subset of the set of output symbols Y and the second element is a positive real number less than unity, representing a probability measure.

In the case of generalized stochastic automata it is possible to represent the output function by a deterministic application depending only on the current state. It was shown that for any generalized stochastic automaton there exists an equivalent Moore stochastic automaton with deterministic output function. [7]

By stochastic automaton with imprecise output function we mean that there is inexact information about the output symbol corresponding to a new internal state. This inexactity consists of both imprecision and uncertainty. For example, for state s_j, the application μ can be of the form:

$$\mu(s_j) = \{<\{y_1, y_4\}, p_{1j}>, <\{y_3\}, p_{2j}>, \ldots, <\{y_k, \ldots, y_m\}, p_j^r>\},$$

where $y_i \in Y$ for all indexes i.

By $<\{y_1, y_4\}, p_{1j}>$ we express the fact that, in state s_j, the automaton may have as output one of the symbols y_1 or y_4 with probability p_j^1. We do not know anything about the chances of y_1 beeing the output prefered to the output y_4, but we know that the choice of the set $\{y_1, y_4\}$ is made with probability p_j^j. Analogously, p_j^r represents the probability to have as output one of the symbols $y_k, \dots, y_m$ without knowing anything about how the symbol will be choosen from the others in the subset to which it belongs.

We may suppose that all values p_j^i are positive values without restricting generality. Note that p_j^i, as probabilities assigned to subsets of Y, are values of a probability distribution on the power set of Y. That means that p_j^i are values of a basic probability assignment on Y. We may use for $\mu(s_j)$ the notation:

$$\mu(s_j) = \{<A_k, m_j(A_k)> | A_k \in \mathcal{P}(Y) \wedge m_j(A_k) > 0\} \tag{5}$$

for $k = 1, \dots, n_j$, where $\sum_{A_j \subseteq Y} m_j(A_k) = 1$ and $m(\emptyset) = 0$.

Note that the set Y_j of possible output symbols corresponding to state s_j is defined using the focal elements of the body of evidence defined on Y, focal elements which correspond to the basic probability assignment m_j. So, the definition says that for each internal state of the automaton there is a probability distribution on the power set of Y, the set of output symbols.

Proposition 2.2. *Let A be a stochastic automaton with imprecise output function as described in Definition 2.1. If the definition of the application μ is based on a basic probability assignment on the singletons (sets which consist only of one element) of Y, then the automaton is a generalized stochastic automaton as it was described in Definition 1.2.*

Proof. Let $A_j^i = \{y_i\}$ for all $i = 1, \dots, r$ and $s_j \in S$. We supposed that all $p_j^i > 0$. We can add all the output symbols of Y which were not mentioned in subsets A_j^i, and we can put $p_j^i = 0$ for all those output symbols.

It is obvious that the relations (4) hold.

In that case, from [4] we know that the values p_j^i represent the values of a probability distribution in classical sense. It is trivial to show that under these circumstances, the automaton is a generalized stochastic automaton.

Example 2.3. *Let $A =< \{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}, \{s_1, s_2, s_3\}, \{P\}, \mu, (1, 0, 0)>$ be a stochastic automaton with imprecise output function, where the transition probabilities are expressed in the form:*

$$P(x_1) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad P(x_2) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \quad P(x) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

and the output function is defined by:

$$\mu(s_1) = \{<\{y_2\}, 0.15>, <\{y_1, y_2\}, 0.85>\}$$

$$\mu(s_2) = \{<\{y_1, y_2\}, 0.2>, <\{y_2, y_3\}, 0.2>, <\{y_3\}, 0.6>\}$$
$$\mu(s_3) = \{<\{y_1\}, 0.5>, <\{y_2\}, 0.5>\}$$

Note that the initial state is s_1 and $\sum_i p_j^i = 1$, $p_j^i > 0$ for all $i = 1, \ldots, 3$.

Definition 2.4. *Let A be an automaton as in 2.1 The pair of values:*

(6)
$$CL_s(A_k) = \{<bel_s(A_k), pls_s(A_k)> | bel_s(A_k), pls_s(A_k) \in [0,1] \wedge bel_s(A_k) \leq pls_s(A_k)\}$$

is called confidence level and is defined for all subsets $A_k \in \mathcal{P}(Y)$ of output symbols and for any state $s \in S$.

In the above definition *bel* and *pls* denote the belief and the plausibility values respectively, values corresponding to the subset A_k of possible output symbols. These values depend on the current state s of the automaton.

We may say that CL represents the degree of belief, or the degree of confidence that we have in the source of informations concerning the subsets of possible output symbols.

Starting from the values of the basic probability assignment, which has positive values for some subsets A_k, we may establish for any subset of output symbols the lower limit (the belief value) and the upper limit (the plausibilty value) of the confidence that we have assigned to that subset.

The upper and the lower limits are obtained according to the relations (3), where we use one basic probability assignment on Y for a certain moment. Note that there may exist many basic probability assignments on Y for the same state s. They express the available evidence for some subsets of Y at a certain moment.

To compute the values of $\mathrm{CL}_s(A_k)$, for any $A_k \subseteq Y$, we need the following formulas:

(7)
$$\mathrm{bel}_s(A_k) = \sum_{B \subseteq A_k} m_s(B) = \sum_{B \subseteq A_k} \mu(s; B),$$
$$\mathrm{pls}_s(A_k) = \sum_{B \cap A_k \neq \emptyset} m_s(B) = \sum_{B \cap A_k \neq \emptyset} \mu(s; B)$$

With $\mu(s; B)$ we denote the values p_j^i corresponding to the subset B which, from all pairs defining $\mu(s)$, correspond to the sum conditions in the formula.

Example 2.5. *Let A be the automaton described in Example 2.5. Let's compute the confidence levels for the subset $K = \{y_2\}$. We obtain:*

$$\begin{aligned} CL_{s_1}(K) &= \{<0.15, 1>\} \\ CL_{s_2}(K) &= \{<0, 0.4>\} \\ CL_{s_3}(K) &= \{<0.5, 0.5>\} \end{aligned}$$

From (7) we have

$$\begin{aligned} &bel_{s_1}(K) = 0.15, \quad pls_{s_1}(K) = 0.15 + 0.85 = 1, \\ &bel_{s_2}(K) = 0, \quad pls_{s_2}(K) = 0.2 + 0.2 = 0.4, \\ &bel_{s_3}(K) = 0.5, \quad pls_{s_3}(K) = 0.5 \end{aligned}$$

In the case of a stochastic automaton with imprecise output function, we admit to consider each possible output symbol together with its CL_s, computed according to the current basic probability assignment, which means the current amount of evidence on the outputs of the automaton in each internal state s.

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