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# A Note on the Isomorphic Representation of Nondeterministic Nilpotent Automata

ILDIKÓ SZÉKELY

**Abstract.** It is known that if an automaton of $s$ states can be embedded isomorphically into a direct product of automata having fewer states than $s$, then the automaton considered can be embedded isomorphically into a direct product of two automata having fewer states than $s$. In particular, this statement is also valid for classes of automata which are closed under homomorphism. In this paper, it is shown that this observation does not hold for nondeterministic automata in general.

## 1. Introduction

Let $\mathcal{M}$ be an arbitrary class of automata which is closed under the homomorphism, and let $\mathbf{A}$ be an element of $\mathcal{M}$ with $s$ states. From a theorem of Birkhoff (see [4]), we can conclude that if $\mathbf{A}$ can be embedded isomorphically into a direct product of automata from $\mathcal{M}$ having fewer states that $s$, then $\mathbf{A}$ can be embedded isomorphically into a direct product of two automata from $\mathcal{M}$ having fewer states than $s$. Here, we show that this statement is not true for nondeterministic automata in general. Namely, we consider the class of nondeterministic nilpotent automata which is closed under the homomorphism, and construct a nondeterministic nilpotent automaton with 6 states such that this automaton can be embedded isomorphically into a direct product of three nondeterministic nilpotent automata having fewer states than 6, but cannot be embedded isomorphically into a direct product of two nondeterministic nilpotent automata having fewer states than 6. In this way, we prove that not every nondeterministic nilpotent automata which can be embedded isomorphically into a direct product of nondeterministic nilpotent automata having fewer states can be also embedded into a direct product of two nondeterministic nilpotent automata having fewer states.

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1991 *Mathematics Subject Classification.* F.1.1 **[Computation by Abstract Devices]**: Models of Computation - *Nondetermination*; F.4.3 **[Mathematical Logic and Formal Languages]**: Formal Languages - *Classes defined by grammars or automata* .

## 2. Preliminaries

The notion of the nilpotent automaton (see e.g. [1], [5] and [8],) can be extended to nondeterministic automataa, and it is an interesting question whether a given nondeterministic nilpotent automaton can be embedded isomorphically into a direct product of nondeterministic nilpotent automata with fewer states than the considered one. This problem is solved in [9]. Studies on representation of nondeterministic automata under stronger compositions than the direct product can be found in the works [2], [3], [6] and [7]. Here, we deal with the simplest composition, the direct product. First of all, we recall the basic notions and some results given in [9]. If $X$ and $A$ are nonempty finite sets and every $x \in X$ is realized in $A$ as a binary relation $x^{\mathbf{A}}$ then the system $\mathbf{A} = (X, A)$ is called a *nondeterministic automaton*. $X$ is the set of *input signs* and $A$ is the set of *states*. For a given $x \in X$ and $a \in A$, the set $\{a' | a' \in A \text{ and } ax^{\mathbf{A}}a'\}$ is denoted by $ax^{\mathbf{A}}$. It can be regarded as the set of all states in which the automaton transits if the current state is $a$ and the input sign is $x$. For a $p \in X^+$, the binary relation $ap^{\mathbf{A}}$ can be defined in the following way:

$$ap^{\mathbf{A}} = \begin{cases} \bigcup\limits_{b \in aq^{\mathbf{A}}} bx^{\mathbf{A}} & \text{if } p = qx \text{ where } x \in X \text{ and } q \in X^+, \\ ax^{\mathbf{A}} & \text{if } p = x \text{ where } x \in X. \end{cases}$$

If $M \subseteq A$, $x \in X$, $p \in X^+$, then $Mx^{\mathbf{A}} = \bigcup\limits_{a \in M} ax^{\mathbf{A}}$ and $Mp^{\mathbf{A}} = \bigcup\limits_{a \in M} ap^{\mathbf{A}}$.

Let now $\mathbf{A} = (X, A)$ and $\mathbf{B} = (X, B)$ be two nondeterministic automata. It is said that $\mathbf{B}$ is a *subautomaton* of $\mathbf{A}$ if $B \subseteq A$ and $x^{\mathbf{B}}$ is the restriction of $x^{\mathbf{A}}$ to $B \times X$, i.e. $ax^{\mathbf{B}} = ax^{\mathbf{A}} \cap B$ holds, for all $a \in A$ and $x \in X$. A mapping $\mu$ of $A$ into $B$ is called *homomorphism*, if $\mu(ax^{\mathbf{A}}) = \mu(a)x^{\mathbf{B}}$ is valid, for all $a \in A$ and $x \in X$. If $\mu$ is also a one-to-one mapping onto $B$, then $\mu$ is called *isomorphism*. In this latter case, we say that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic.

Let $k$ be an arbitrary positive integer and $\mathbf{A}_r = (X_r, A_r)$, $r = 1, \ldots, k$ be nondeterministic automata. By the *direct product* of $\mathbf{A}_r = (X_r, A_r)$, $r = 1, \ldots, k$, we mean the automaton $\mathbf{A} = (X, A)$ where $A = A_1 \times \cdots \times A_k$ and for every $a = (a_1, \ldots, a_k) \in A$ and $x \in X$, $ax^{\mathbf{A}} = (a_1, \ldots, a_k)x^{\mathbf{A}} = a_1x^{\mathbf{A}_1} \times \cdots \times a_kx^{\mathbf{A}_k}$. We will denote by $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ the direct product of $\mathbf{A}_1, \ldots, \mathbf{A}_k$. A nondeterministic automaton $\mathbf{A}$ *can be embedded isomorphically* into the direct product of $\mathbf{A}_1, \ldots, \mathbf{A}_k$ if $\mathbf{A}$ is isomorphic to a subautomaton $\mathbf{B}$ of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$.

A nondeterministic automaton $\mathbf{A} = (X, A)$ is called *complete* if $ax^{\mathbf{A}} \neq \emptyset$ holds, for all $a \in A$ and $x \in X$. As in [9], by nondeterministic automaton we will always mean complete nondeterministic automaton having at least two states. In this note, we deal with a special class of nondeterministic automata, the class of nondeterministic nilpotent automata defined as follows.

A nondeterministic automaton $\mathbf{A} = (X, A)$ is called *nilpotent* if there exists a positive integer $n$ such that $Ap^{\mathbf{A}} = \{a_0\}$ is valid, for all $p \in X^+$ with $|p| \geq n$, where $|p|$ denotes the length of the word $p$. The distinguished state $a_0$ is called the *absorbent state* of $\mathbf{A}$.

For the class of the nondeterministic nilpotent automata the following statement can be easily proved.

**Lemma 1.** *The class of nondeterministic nilpotent automata is closed under the homomorphism.*

Let us define the following relation on $A$: $a \leq b$ if and only if $a = b$ or there is a $p \in X^+$ such that $b \in ap^{\mathbf{A}}$. It is easy to see that the introduced relation is a partial ordering on $A$. If one of the relations $a \leq b$ or $b \leq a$ is valid for some $a, b \in A$, then $a$ and $b$ are called *comparable*. Otherwise, we say that they are *incomparable* and we denote it by $a \bowtie b$.

It is obvious that the absorbent state $a_0$ is the greatest element in $(A, \leq)$. Furthermore, since $|A| \geq 2$, there must be at least one $b_0 \neq a_0 \in A$ such that $b_0$ is a maximal element in $(A \setminus \{a_0\}, \leq)$. A direct consequence of the nilpotency of $\mathbf{A}$ is that $a_0$ and $b_0$ satisfy the identity $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$, for all $x \in X$. If there is exactly one pair of different states $a \neq b$ in $A$ satisfying $ax^{\mathbf{A}} = bx^{\mathbf{A}}$, for all $x \in X$, then these states have to be $a_0$ and $b_0$. We use the following sentence to express this : "$\mathbf{A}$ has exactly one pair of different states $a_0, b_0$ for which $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ holds for all $x \in X$".

The following property shows that no "loops" or "circuits" may appear on the states of a nondeterministic nilpotent automaton except for the absorbent state.

**Lemma 2.** *If $\mathbf{A} = (X, A)$ is a nondeterministic nilpotent automaton with the absorbent state $a_0$, then $a \notin ap^{\mathbf{A}}$ holds, for all $a \in A \setminus \{a_0\}$ and $p \in X^+$.*

From the definitions we can immediately see that if $\mathbf{A}_1, \ldots, \mathbf{A}_k$ are nondeterministic nilpotent automata with the absorbent states $a_1^0, \ldots, a_k^0$, respectively, then their direct product is a nondeterministic nilpotent automaton with the absorbent state $(a_1^0, \ldots, a_k^0)$. Furthermore, every complete subautomaton of the direct product is nilpotent with the absorbent state $(a_1^0, \ldots, a_k^0)$. This also yields that if a nondeterministic nilpotent automaton $\mathbf{A} = (X, A)$ with the absorbent state $a_0$ can be embedded under the isomorphism $\mu$ into $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$, then $\mu(a_0) = (a_1^0, \ldots, a_k^0)$.

**Lemma 3.** *Assume that $\mathbf{A} = (X, A)$ is a nondeterministic nilpotent automaton that has exactly one pair of different states $a_0, b_0$ for which $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ holds, for all $x \in X$. Let $\mathbf{A}_r = (X, A_r), r = 1, \ldots, k, (k \geq 2)$ be nondeterministic nilpotent automata with the absorbent states $a_1^0, \ldots, a_k^0$, respectively, and let $\mu : A \to B \subseteq A_1 \times \cdots \times A_k$ be an isomorphism embedding $\mathbf{A}$ into $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k$. If we denote by $(b_1^0, \ldots, b_k^0)$ the image of $b_0$ under $\mu$ and by $I$ the following set of indices $I = \{i \in \{1, \ldots, k\} | a_i^0 \neq b_i^0\}$, then the components $a_i^0, b_i^0$ with $i \in I$ may appear in no other elements of $\mathbf{B}$ but $\mu(a_0)$ and $\mu(b_0)$.*

Now, we recall a result from [9] giving a sufficient condition for the isomorph[ic] embedding of a nondeterministic nilpotent automata.

**Lemma 4.** *Let $\mathbf{A} = (X, A)$ be a nondeterministic nilpotent automaton with* $|A| \geq$ *4 that has exactly one pair of different states $a_0, b_0$ for which $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ holds for all $x \in X$. If there is a natural number $k \geq 2$ and there are $c, e_1, \ldots, e_k \in A$ with $c \bowtie e_r, r = 1, \ldots, k$, such that*

$$\forall x \in X \left(\exists i \in \{1, \ldots, k\} : e_i x^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq c x^{\mathbf{A}} \cap \{a_0, b_0\}\right) \text{ holds,}$$

*then $\mathbf{A}$ can be embedded isomorphically into a direct product of $k+1$ nondeterministic nilpotent automata having fewer states than $|A|$.*

## 3. A nondeterministic nilpotent automaton having no subdirect decomposition of two factors

Let us consider the following automaton: $\mathbf{A} = (X, A)$, where $X = \{x, y\}$, $A = \{a_0, b_0, d, c, e, f\}$ and the binary relations $x^{\mathbf{A}}$ and $y^{\mathbf{A}}$ are given by the following table:

| | $x^{\mathbf{A}}$ | $y^{\mathbf{A}}$ |
|---|---|---|
| $a_0$ | $\{a_0\}$ | $\{a_0\}$ |
| $b_0$ | $\{a_0\}$ | $\{a_0\}$ |
| $d$ | $\{b_0\}$ | $\{b_0\}$ |
| $c$ | $\{b_0, d\}$ | $\{b_0\}$ |
| $e$ | $\{b_0\}$ | $\{a_0, d\}$ |
| $f$ | $\{a_0\}$ | $\{b_0, d, e\}$ |

The transition graph of $\mathbf{A}$ is depicted in Figure 1. We will show that the automaton given above can be embedded isomorphically into a direct product of three nondeterministic nilpotent automata having fewer states than $|A|$ but cannot be embedded into a direct product of two nondeterministic nilpotent automata having fewer states than $|A|$.

**Theorem 1.** *The automaton $\mathbf{A}$ can be embedded isomorphically into a direct product of three nondeterministic nilpotent automata having fewer states than $|A|$.*

**Proof.** It is obvious that $\mathbf{A}$ is a nondeterministic nilpotent automaton with the absorbent state $a_0$ and that $a_0$ and $b_0$ form the only pair of different states of $A$ satisfying $a_0 x^{\mathbf{A}} = b_0 x^{\mathbf{A}}$ and $a_0 y^{\mathbf{A}} = b_0 y^{\mathbf{A}}$. Moreover, we have the state $c$, which is incomparable with $e, f$ for which the statements $e x^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq c x^{\mathbf{A}} \cap \{a_0, b_0\}$ and $f y^{\mathbf{A}} \cap \{a_0, b_0\} \subseteq c y^{\mathbf{A}} \cap \{a_0, b_0\}$ hold.

Hence, the conditions of Lemma 4 are satisfied, and thus, $\mathbf{A}$ can be embedded isomorphically into a direct product of three nondeterministic nilpotent automata. □

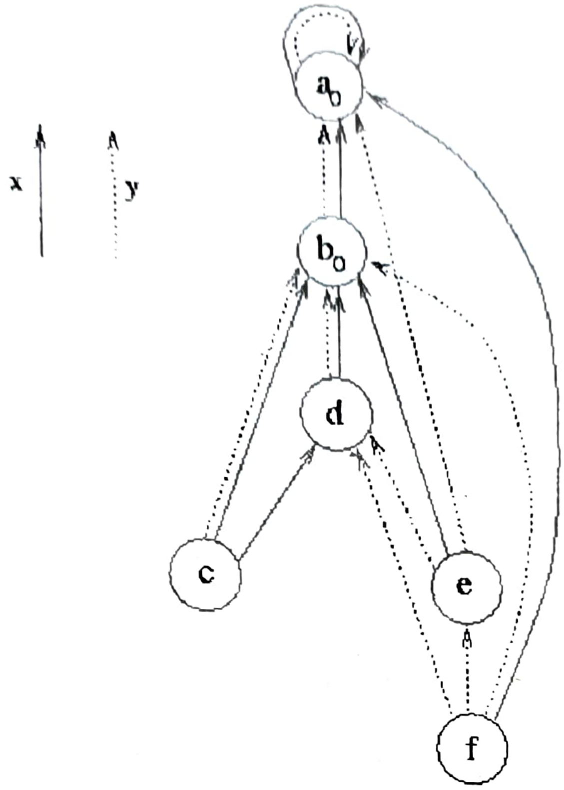


FIGURE 1. Automaton **A**.

**Theorem 2.** *The automaton* **A** *given above cannot be embedded isomorphically into a direct product of two nondeterministic nilpotent automata having fewer states than* $|A|$.

**Proof.** For verification by contradiction, let us assume that **A** can be embedded isomorphically into a direct product of two nondeterministic nilpotent automata having fewer states than $|A| = 6$. Thus, there exist nondeterministic nilpotent automata $\mathbf{A}_r = (X, A_r)$ with $|A_r| < |A| = 6$, $r = 1, 2$ and there also exists an isomorphism $\mu : A \to B \subseteq A_1 \times A_2$ that embeds **A** into $\mathbf{A}_1 \times \mathbf{A}_2$. Let **B** denote the isomorphic image of **A** in $\mathbf{A}_1 \times \mathbf{A}_2$. Furthermore, let $\mu(a_0) = (a_1^0, a_2^0)$ and $\mu(b_0) = (b_1^0, b_2^0)$. Since $a_0 \neq b_0$, $(a_1^0, a_2^0) \neq (b_1^0, b_2^0)$. Therefore, $a_1^0 \neq b_1^0$ or $a_2^0 \neq b_2^0$. Without loss of generality, it can be supposed that $a_1^0 \neq b_1^0$. Furthermore, the set of the first components of $\mu(c), \mu(d), \mu(e), \mu(f)$ is disjoint to $\{a_1^0, b_1^0\}$ from Lemma 3. We use the following notations: $\mu(c) = (c_1, c_2)$, $\mu(d) = (d_1, d_2)$, $\mu(e) = (e_1, e_2)$ and $\mu(f) = (f_1, f_2)$. Now we distinguish two cases depending on $a_2^0$ and $b_2^0$, namely $a_2^0 = b_2^0$ or $a_2^0 \neq b_2^0$.

**Case 1.** Let $a_2^0 = b_2^0$. Then $\mu(a_0) = (a_1^0, a_2^0)$ and $\mu(a_0) = (b_1^0, a_2^0)$. Let us denote by $\overline{a}_1$ the state which occurs in at least two different pairs of $B$ on the first position. The existence of $\overline{a}_1$ follows from the fact that $|A_1| < |A| = 6$. Then

$\overline{a}_1 \notin \{a_1^0, b_1^0\}$. According to this and to Lemma 2 saying that in a nondeterministic nilpotent automaton no state may have "loops" or "circuits" except for the absorbent state, the ancestors with respect to $\mu$ of the pairs of $B$ which have $\overline{a}_1$ on the first position are incomparable in $\mathbf{A}$. Since the incomparable pairs in $\mathbf{A}$ are $c \bowtie e$ and $c \bowtie f$, we have $\overline{a}_1 = c_1$. The two possible cases are when $e_1$ is $c_1$ or when $f_1$ is $c_1$. $c_1 = e_1$ and $c_1 = f_1$ cannot be satisfied in the same time, because $e$ and $f$ are comparable in $\mathbf{A}$.

Suppose first that $\mu(e) = (c_1, e_2)$. From $\mu(ey^{\mathbf{A}}) = \{(a_1^0, a_2^0), (d_1, d_2)\}$ and $\mu(e)y^{\mathbf{B}} = c_1 y^{\mathbf{A}_1} \times e_2 y^{\mathbf{A}_2} \cap B$, we have $a_1^0 \in c_1 y^{\mathbf{A}_1}$. In the same time, from $\mu(cy^{\mathbf{A}}) = \{(b_1^0, a_2^0)\}$ and $\mu(c)y^{\mathbf{B}} = c_1 y^{\mathbf{A}_1} \times c_2 y^{\mathbf{A}_2} \cap B$, we can conclude $a_2^0 \in c_2 y^{\mathbf{A}_2}$. This yields $(a_1^0, a_2^0) \in c_1 y^{\mathbf{A}_1} \times c_2 y^{\mathbf{A}_2} \cap B$, which implies immediately $(a_1^0, a_2^0) \in (c_1, c_2)y^{\mathbf{B}}$ and $\mu(a^0) \in \mu(c)y^{\mathbf{B}}$. Since $\mu$ is an isomorphism, we must also have $a_0 \in cy^{\mathbf{A}}$, which is not true by the definition of the automaton $\mathbf{A}$.

Suppose now that $\mu(f) = (c_1, f_2)$. Then, $\mu(fx^{\mathbf{A}}) = \{(a_1^0, a_2^0)\}$ and $\mu(f)x^{\mathbf{B}} = c_1 x^{\mathbf{A}_1} \times f_2 x^{\mathbf{A}_2} \cap B$ yield $a_1^0 \in c_1 x^{\mathbf{A}_1}$. On the other hand, we have $\mu(cx^{\mathbf{A}}) = \{(b_1^0, a_2^0), (d_1, d_2)\}$ and $\mu(c)x^{\mathbf{B}} = c_1 x^{\mathbf{A}_1} \times c_2 x^{\mathbf{A}_2} \cap B$, which imply $a_2^0 \in c_2 x^{\mathbf{A}_1}$. Consequently, $(a_1^0, a_2^0) \in (c_1, c_2)x^{\mathbf{B}}$ and thus $\mu(a^0) \in \mu(c)x^{\mathbf{B}}$, resulting in $a_0 \in cx^{\mathbf{A}}$, which is also a contradiction.

**Case 2.** Let $a_2^0 \neq b_2^0$. Then $\mu(a_0) = (a_1^0, a_2^0)$ and $\mu(b_0) = (b_1^0, b_2^0)$. Since $A_1$ and $A_2$ have fewer states than $|A|$, there has to be an $\overline{a}_1 \in A_1$ which appears in at least two different pairs of $B$ in the first component and there has to be an $\overline{a}_2 \in A_2$ which appears in at least two different pairs of $B$ in the second component, furthermore $\overline{a}_r \notin \{a_r^0, b_r^0\}, r = 1, 2$. The mentioned pairs of $B$ are the images of incomparable states of $\mathbf{A}$. Since the incomparable pairs of $\mathbf{A}$ are $c \bowtie e$ and $c \bowtie f$, $\mu(c) = (\overline{a}_1, \overline{a}_2)$. We have to discuss the following four possibilities: first $\mu(e) = (c_1, e_2)$ and $\mu(f) = (f_1, c_2)$, second $\mu(e) = (e_1, c_2)$ and $\mu(f) = (c_1, f_2)$, third $\mu(e) = (c_1, c_2)$ and fourth $\mu(f) = (c_1, c_2)$. In the first two cases the proof is similar to the ones given in Case 1.

If $\mu(e) = (c_1, e_2)$ and $\mu(f) = (f_1, c_2)$, then $\mu(ey^{\mathbf{A}}) = \mu(e)y^{\mathbf{B}}$ yields $d_1 \in c_1 y^{\mathbf{A}_1}$ and $\mu(fy^{\mathbf{A}}) = \mu(f)y^{\mathbf{B}}$ yields $d_2 \in c_2 y^{\mathbf{A}_2}$, which lead to the contradiction $d \in cy^{\mathbf{A}}$.

If we assume $\mu(e) = (e_1, c_2)$ and $\mu(f) = (c_1, f_2)$, then we have $\mu(fy^{\mathbf{A}}) = \mu(f)y^{\mathbf{B}}$, which yields $d_1 \in c_1 x^{\mathbf{A}_1}$. On the other hand $\mu(ey^{\mathbf{A}}) = \mu(e)y^{\mathbf{B}}$ implies $d_2 \in c_2 y^{\mathbf{A}_2}$. These two statements lead to $d \in cy^{\mathbf{A}}$, which is in contradiction with the definition of automaton $\mathbf{A}$.

If $\mu(e) = (c_1, c_2)$, then $\mu(c) = \mu(e)$ and since $\mu$ is an isomorphism, this yields $c = e$, which is a contradiction. The proof is similar in the fourth case.

Thus, we have shown that the starting assumption of our proof leads to a contradiction in all cases. This completes the proof of Theorem 2. □

## References

[1] F. Gécseg, *Products of automata*, EATCS Monographs on Theoretical Computer Science, Springer Verlag, 1986

[2] F. Gécseg, B. Imreh, On completeness of nondeterministic automata, *Acta Math.Hungar.* **68** (1995), 151-159.

[3] F. Gécseg, B. Imreh, On the cube product of nondeterministic automata, *Acta Sci.Math.(Szeged)* **60** (1995), 321-327.

[4] G. Grätzer, *Universal algebra*, 2-nd edition Springer Verlag (New-York-Berlin-Heidelberg-Tokyo), 1986

[5] B. Imreh, On finite nilpotent automata, *Acta Cybernetica* **5** (1981), 281-293.

[6] B. Imreh, Compositions of nondeterministic automata, Symposium on Semigroups, Formal Languages and Computer Science, *RIMS, Kokyuroku* **960** (1996), 44-53.

[7] B. Imreh, On $\alpha_i$-Product of Nondeterministic Automata *Algebra Colloquium* **4:2** (1997), 195-202.

[8] L. N. Sevrin, On some classes of abstract automata, *Uspekhi matem. nauk* **17:6** (108) (1962), 219-223.

[9] I. Székely, Isomorphic representation of nondeterministic nilpotent automata, to appear in *Acta Cybernetica* (1998).

Department of Informatics, József Attila University, Árpád tér 2, H-6720 Szeged, Hungary