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SEMANTIC TABLEAUX FOR SEMI-NORMAL DEFAULTS THEORIES

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Abstract. This article presents the semi-normal default theories and the classes of semi-normal default theories which have allways extensions. A proof method based on semantic tableaux for semi-normal default theories is provides and some interesting considerations related to particular classes of theories are exposed.

1. Introduction

Default logic was introduced by Reiter [2] and provides a formalism for an important part of human reasoning, the non- monotonic reasoning. Non-monotonic reasoning means infer conclusions from incomplete information, conclusions which may be invalidated by adding new facts. Default logic is widely used for declarative representations of problems in a variety of areas, including diagnostic reasoning, theory of speech acts, natural language processing, inheritance hierarchies with exceptions.

Definition 1.1. *Let L be a first-order language. A* ***default theory*** $\Delta = (D, W)$ *consists of a set W of closed formulas in first-order logic and a set D of defaults* $d = \frac{\alpha(x):\beta_1(x),\ldots,\beta_m(x)}{\gamma(x)}$, *where*

-$\alpha(x), \beta_1(x), \ldots, \beta_m(x), \gamma(x)$ *are well formed formulas in first-order logic;*
-$\alpha(x)$ *is called the* ***prerequisite*** *of the default d;*
-$\beta_1(x), \ldots, \beta_m(x)$ *are called* ***justifications*** *of the default d;*
-$\gamma(x)$ *is called the* ***consequent*** *of the default d.*

A default is interpreted as follows:"if I believe α and I have no reason to believe that one of the β_i is false, then I can believe γ. Defaults are inference rules which model the non-monotonic reasoning.

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We denote
$PRE(D) = \bigcup\{\alpha | \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D\}$;
$CONS(D) = \bigcup\{\gamma | \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D\}$ and
$JUST(D) = \bigcup\{\beta_1,\ldots,\beta_m | \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D\}$

The set of defaults D induces an extension on W. Intuitively, an extension is a maximal set of formulas that is deducible from W using the defaults in D and the inference rules from classical logic.

Definition 1.2. *Let $\Delta = (D, W)$ be a closed default theory. For any set of closed formulas $S \subseteq L$, let $\Gamma(S)$ be the smallest set satisfying the following properties: (i) $W \subseteq \Gamma(S)$;*
(ii) $Th(\Gamma(S)) = \Gamma(S)$, where $Th(S) = \{P | S \vdash P\}$;
(iii) if $\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D$ and $\alpha \in \Gamma(S)$ and $\neg\beta_1, \ldots, \neg\beta_m \notin S$ then $\gamma \in \Gamma(S)$.
*A set of closed formulas $E \subseteq L$ is an **extension** for Δ if and only if $\Gamma(E) = E$, i.e. E is a fixed point of the operator Γ.*

The default theories can have zero, one or more extensions.

Definition 1.3. *The set of **generating defaults** of an extension E of a closed default theory is*
$GD(E, \Delta) = \{d \in D | d = \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma}$ *and* $\alpha \in E$ *and* $\neg\beta_1, \ldots, \neg\beta_m \notin E\}$

Theorem 1.4. *Let E be an extension of the closed default theory $\Delta = (D, W)$. Then $E = Th(W \cup CONS(GD(E, \Delta)))$.*

2. The semi-normal theories

Definition 2.1. *A **semi-normal default** is a default of the form: $\frac{\alpha:\beta\wedge\gamma}{\beta}$. A default theory is called **semi-normal** if all the defaults of the theory are semi-normals.*

Semi-normal default theories are not as well behaved as normal default theories, they may have zero, one or many extensions.
Etherington [1] constructed the class of ordered semi-normal default theories, which have always an extension.

Definition 2.2. *A **clausal default theory** is a default theory $\Delta = (D, W)$ in which any sentence in $W \cup PRE(D) \cup JUST(D) \cup CONS(D)$ is logically equivalent to a conjunction of disjunctions of literals.*

Definition 2.3. *Let $\Delta = (D, W)$ be a clausal semi-normal default theory. The partial relations $\ll$ and $\lll$ on $LITERALS \times LITERALS$, are defined as follows:*
(1) If $\alpha \in W$, then $\alpha = \alpha_1 \vee \ldots \vee \alpha_n$ for some $n \geq 1$. For all $\alpha_i, \alpha_j \in \{\alpha_1, \ldots, \alpha_n\}$, if $i \neq j$, let $\neg\alpha_i \lll \alpha_j$
(2) If $\delta \in D$, then $\delta = \frac{\alpha:\beta\wedge\gamma}{\beta}$. Let $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_r, \beta = \beta_1 \wedge \ldots \wedge \beta_s$, and

$\gamma = \gamma_1 \wedge \ldots \wedge \gamma_t$, *and* $\alpha_i = \alpha_{i,1} \vee \ldots \vee \alpha_{i,m_i}$ $\beta_j = \beta_{j,1} \vee \ldots \vee \beta_{j,n_j}$ $\gamma_k = \gamma_{k,1} \vee \ldots \vee \gamma_{k,p_k}$
Then

(i) $\alpha_{i,l} \lll \beta_{j,q}$ *for all* $i \in \{1, .., r\}, l \in \{1, ..., m_i\}, j \in \{1, ..., s\}, q \in \{1, ..., n_j\}$.
(ii) *if* $\gamma_{k,l} \notin \{\beta_{1,1}, \ldots \beta_{s,n_s}\}$ *let* $\neg\gamma_{k,l} \ll \beta_{j,q}$.
(iii) *if* $l \neq k$ *let* $\neg\beta_{j,l} \lll \beta_{j,k}$.

(3) *Let L,L',L" be literals, the the expected transitivity relantionships hold for* $\ll$ *and* $\lll$:

(i) *if* $L \lll L'$ *and* $L' \lll L"$ *then* $L \lll L"$.
(ii) *if* $L \ll L'$ *and* $L' \ll L"$ *then* $L \ll L"$.
(iii) *if* $L \ll L'$ *and* $L' \lll L"$ *or* $L \lll L'$ *and* $L' \ll L"$ *then* $L \ll L"$.

Definition 2.4. *A semi-normal defaulf theory is said to be **ordered** if and only if there is no literal L, such that* $L \ll L$.

Theorem 2.5. *An ordered semi-normal default theory has at least one extension.*

Definition 2.6. *A **graph** of a propositional semi-normal default theory* $\Delta = (D, W)$, $G = (E, A)$ *is a directed graph with defaults as nodes and whose set of arcs are partitioned into arcs of weight 0 and arcs of weight 1.*
- There is an arc $(d_1, d_2) \in A$ *of weight 0 if there is a literal* $l \in CONS(d_1)$ *such that* $l \in JUST(d_2)$.
- There is an arc $(d_1, d_2) \in A$ *of weight 1 if there is a literal* $\neg l \in CONS(d_1)$ *such that* $\neg l \in JUST(d_2) \cup PRE(d_2) \cup CONS(d_2)$.
- There is an arc $(d_1, d_2) \in A$ *of weight 0 if there is a literal* $l \in CONS(d_1)$ *such that* $\neg l, e \in \omega JUST(d_2) \cup PRE(d_2) \cup CONS(d_2)$.
*An **even** propositional semi-normal default theory, is a default theory with a graph that contains only cycles of even weight.*

Theorem 2.7. *Every even propositional semi-normal default theory has an extension.*

Theorem 2.8. *If* (D, W) *is an even default theory, then* (D', W'), *where* $D' \subset D$, *and* $W' \subset W$, *is also an even default theory.*

Remark:
The theorem2 and theorem3 provide a sufficient condition for the existance of an extension of a semi-normal default theories. These conditions are not necessary, that is exist semi-normal theories which are neither even, nor ordered, but have extensions.

Example 2.9. *Let* $\Delta = (D = \{d_1, d_2, d_3\}, \emptyset)$ *be a semi-normal default theory, where* $d_1 = \frac{:p \wedge \neg q}{\neg q}$, $d_2 = \frac{:q \wedge \neg r}{\neg r}$, $d_3 = \frac{:r \wedge \neg p}{\neg p}$.

This theory has no extension, is not ordered because $\neg p \ll \neg q \ll \neg r \ll \neg p$, thus $\neg p \ll \neg p$, and is even.

Example 2.10. *The semi-normal default theory* $\Delta' = (D, \{p\})$ *is also an even theory, is not ordered, but has an extension* $E = Th(\{p, \neg q\})$.

3. The semantic tableaux for first order logic

The semantic tableaux is a refutation proof method which try to construct all the models of a formula, and therefore is well adapted for producing extensions of default theories. In [3] is introduced and we present here, a new conception of semantic tableaux as a proof method for first order logic.

Definition 3.1. *We denote TP the* ***tableaux prover*** *which is defined recursively as a mapping between sets of formulae and sets of literals as follows:*
$TP(M) = TP(M' \cup \{p\})$ *if* $\neg\neg p \in M$ *and* $M' = M \backslash \{\neg\neg p\}$
$TP(M) = TP(M' \cup \{p\} \cup \{q\})$ *if* $p \wedge q \in M$ *and* $M' = M \backslash \{p \wedge q\}$
$TP(M) = TP(M' \cup \{p\}) \cup TP(M' \cup \{q\})$, *if* $p \vee q \in M$ *and* $M' = M \backslash \{p \vee q\}$
$TP(M) = TP(M \cup \{A[c]\})$, *if* $(\exists x)A(x) \in M$, *where c is a new parameter*
$TP(M) = TP((M \cup \{A[c]\})$, *if* $(\forall x)A(x) \in M$, *where c is a any parameter*
$TP(M) = \{M\}$, *if M is a set of literals.*

Definition 3.2. *A set of literals is* ***closed*** *if it contains two opposite literals. A set of sets of literals is* ***closed*** *if each of its elements is closed*

The following theorem asserts the completeness of this proof method.

Theorem 3.3. *F is a theorem if and only if* $TP(\{\neg F\})$ *is closed.*

Remark:
For a formula F, TP(F) corresponds to the disjunctive normal form of F, each disjunct represents a set of models of the formula. The disjunctive normal form of a formula F obtained from TP(F) is denoted $\phi(TP(\{F\}))$. The TP has also many interesting properties concernining manipulation of formulas and sets of formulas.
$\Gamma \otimes \Gamma' = \{X \cup Y | X \in \Gamma \text{ and } Y \in \Gamma'\}$

Theorem 3.4. *Let* Γ *be a finite set of formulas and F a formula.* $F \in Th(\Gamma)$ *if and only if* $TP(\Gamma) \otimes TP(\{\neg F\})$ *is closed.*

Definition 3.5. *A tableau is a* ***subtableau*** *of a tableau T',noted* $T \prec T'$ *if and only if every set of T' contains a set of T.*

Definition 3.6. *An opening O of* $T \otimes T'$ *is called* ***opening of*** $T \otimes T'$ ***in T'*** *if* $O = T \otimes T''$ *and* $T'' \prec T$.

We can obtain a minimal form of TP(M) eliminating closed sets and sets which contain other sets in TP(M). This minimal form is denoted
$TS(M) = TP(M) \backslash \{X | X \in TP(M), X$ is closed, or $Y \in TP(M)$ such that $Y \subset X, Y \neq X\}$.
We have $\phi(TS(M)) \Leftrightarrow \phi(TP(M))$.

3.1. Semantic tableaux for default logic.

Definition 3.7. *Let D be a set of defaults, D' and D"" two subsets of D, we have $D" \prec_\Delta D"$ if and only if $D' = \{d \in D | d \notin D", andPRE(\{d\} \in Th(W \cup CONS(D"))\} \cup D"$.*

The smallest class of defaults with respect to $\prec_\Delta$ is $\emptyset$.

Definition 3.8. *Any sequence of subsets of D with respect to $\prec_\Delta$ is called a* ***grounding sequence*** *of D. Any grounding sequence of D from $\emptyset$ is called a* ***root*** *of D.*

Definition 3.9. *Let $\Delta = (W, D)$ be a default theory. D is* ***grounded in W*** *if and only if $D = R_\Delta(\emptyset, D)$, where*
$R_\Delta(D', D) = \{d \in D \backslash D' | \exists D_1, ..., D_n$ *such that* $D' \prec_\Delta D_1, ..., D_n \prec_\Delta D\}$.

Theorem 3.10. *The set of generating defaults $GD(E, \Delta)$ of an extension E of a default theory $\Delta = (W, D)$ is grounded in W.*

The following theorem gives a necessary and sufficient existence criterion for extensions of a semi-normal default theory.

Theorem 3.11. *A semi-normal default theory $\Delta = (D, W)$ has an extension E, if and only if exists $D' \subseteq D$, D' grounded in W and $E = Th(W \cup CONS(D'))$ and $\forall d \in D, d = \frac{\alpha:\beta\wedge\gamma}{\beta}$ (i) and (ii) hold:*
(i) if $d \in D'$ then $\alpha \in E, \neg\beta \notin E, \neg\gamma \notin E$.
(ii) if $d \notin D'$ then $\alpha \notin E$ or $\neg\beta \in E$ or $\neg\gamma \in E$.

Remark:
- Condition (i) is equivalent to $TP(W) \otimes TP(CONS(D')) \otimes TP(\{\neg\alpha\})$ closed ,$TP(W) \otimes TP(CONS(D')) \otimes TP(\{\beta\})$ open and $TP(W) \otimes TP(CONS(D')) \otimes TP(\{\gamma\})$ open.
- Condition (ii) is equivalent to $TP(W) \otimes TP(CONS(D')) \otimes TP(\{\neg\alpha\})$ open, or $TP(W) \otimes TP(CONS(D')) \otimes TP(\{\beta\})$ closed, or $TP(W) \otimes TP(CONS(D')) \otimes TP(\{\gamma\})$ closed.

The algorithm for constructing extensions for a semi-normal default theory is:
(1) We construct the maximal consistent subsets of $W \cup CONS(D)$ which contain D. These sets can be obtain opening the semantic tableau $TP(W) \otimes TP(CONS(D))$, by eliminating some literals of defaults. A such set D_i coresponds to an opening of semantic tableau.
(2) We must verify the justifications for every $Th(W \cup CONS(D_i))$. This is equivalent with the second part of condition (i) and (ii)
(3) For thoses sets D_i which are obtained from (2) we must verify the preconditions for all defaults in D_i.
The sets obtained from (3) are the generating sets for the extensions of the semi-normal default theory.

Example 3.12. *Let $\Delta = (D = \{d_1, d_2, d_3\}, \emptyset)$ be a semi-normal default theory, where $d_1 = \frac{:p\wedge\neg q}{p}$, $d_2 = \frac{:q\wedge\neg s}{q}$, $d_3 = \frac{:\neg p\wedge(r\rightarrow s)}{r\rightarrow s}$.*

TP(CONS(D))=$TP(\{\{p, q, \neg r \vee s\}\}=\{\{p, q, \neg r\}, \{p, q, s\}\}$ which is open, therefore may be one extension for this theory.
The preconditions are trivial satisfied. We must verify the justification condition for every default:
- for d_1 : $TP(CONS(D)) \otimes TP(\{\neg q\}) = \{\{p, q, \neg r, \neg q\}, \{p, q, s, \neg q\}\}$ is closed, therefore d_1 is not a generating default.
- for d_2 : $TP(CONS(D)) \otimes TP(\{\neg s\}) = \{\{p, q, \neg r, \neg s\}, \{p, q, s, \neg s\}\}$ is open, therefore d_2 is a generating default for the extension.
- for d_3 : $TP(CONS(D)) \otimes TP(\{\neg p\})$ is open, therefore d_3 is a generating default.
The extension for the semi-normal default theory is $Th(\{q, r \rightarrow s\})$.

Remark: If $W = \{r\}$ then the theory above has no extension.
The algorithm for computing extensions for semi-normal default is very efficient and can be implement very easy.

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