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SYMBOLIC MODELLING OF DYNAMIC EQUATIONS FOR NON-COMPRESSIBLE STATIONARY FLUIDS

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Abstract. This paper deals with automatical implementation of particular theoretical computations which appear in the mechanics of non-compressible stationary fluids. For certain classical types of movements, there were written Mathematica packages which calculate their physical characteristics, such as: velocity, friction tensions, flow, pressure. The results are obtained symbolically, as an expression depending on some variables, but these variables can also be given numerical values, in this case the result will be "partially" or "totally" numerical. Therefore, symbolic computation supports a quick finding of the theoretical desired results, which would otherwise demand a considerable amount of time since they are obtained by differentiations, integrations, and solving quite complicated equations or systems of equations / differential equations.

1. Introduction

Until the last decades, the problems of mathematical physics were solved almost exclusively by numerical methods, therefore no appropriate solution could be found for some of them. Begining with the '70s and especially during the '80s, there took place a huge development of symbolic computation systems for pure mathematics, biology, chemistry, but most of all physics: celestial mechanics, high energy physics, general relativity, electronic optics, molecular physics, fluid mechanics, quantum mechanics [3]. This evolution pursued two directions: building specialized systems for specific domain problems and building applications using the existing general purpose symbolic computation systems (such as MACSYMA, REDUCE, MAPLE, MATHEMATICA) [1]. As some of these systems had an appropiate interface for numerical computations, these types of problems were solved, too.

Dynamic phenomena are described in fluid mechanics by a system of three partial differential equations, which describes the movements on the three axes of

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coordinates, depending on velocity, density, pressure. Considering the complicated form of these equations, which are known as the Navier-Stokes equations, they cannot be solved in the most general case. From the physical point of view, the sequences of computations which lead to the formulas of velocity, fluid flow and friction tensions - for particular situations - are of interest. Here we enumerate such types of movements: newtonian movements between two parallel plates: without pressure gradient and with o moving plate (Couette movement), with pressure gradient between two immobile plates (Poiseuille movement), with a free surface (laminar movement), combined movement (Couette-Poisseuille); movement in cylindrical pipes and between two circular cylinders; non-newtonian parallel plane movements or in cylindrical pipes; non-compressible stationary movements following concentrical circles, between two coaxal cylinders (with a number of special cases), following concurrent lines or between two plane walls.

For all these types of movements there were written Mathematica packages which contain functions for computing the necessary quantities. The built-in Mathematica functions allow to make a simple description of the complicated operations involved in these computations: differentiations, integrations, and solving quite complicated equations or systems of equations / ordinary differential equations. Moreover, working with Mathematica packages, allows a natural and easy extension of the basic capabilities which are available in a Mathematica session. The variables representing arguments for the newly written Mathematica functions can be symbolical or numerical, therefore influencing the result.

In the following paragraphs, we shall present the physical aspects of the problem, the principles used in writing the Mathematica packages, together with a few ideas in respect with the possibility of expanding the computations and an easier interpretation of the result (finding the type of movement based on some characteristics, a graphical representation of the solutions).

2. The problem from the physical point of view

The physical aspects of the problem, together with their mathematical modelling are presented in fluid dynamics [2].

The study of fluid dynamics is a phenomenal one; fluids are considered continuous and deformable media.

The effect of fluid deformation under a shearing force is continuous; fluids flow. The study of real fluids is based on research performed on models, such as the models of perfect fluid (a homogeneous, deformable, non-resisting medium) and perfect viscous fluid.

The perfect viscous fluid, or the newtonian fluid, doesn't immediately react to an action. Its deformation depends on the duration and on the intensity of the solicitation; when the action stops, the deformation doesn't recover and the consumed mechanical work spreads in the whole mass of the fluid as heat. Unless the solicitation modifies, the deformation continues and the deformation velocity remains constant. The newtonian fluid has the property of viscosity - it opposes a resistance to a shearing or compression deformation.

The hypothesis that for newtonian fluids, there is a linear relation between the deformation tensions (shearing tensions $\tau_{ij}, i, j \in \{x, y, z\}, i \neq j$ or compression tensions $\tau_{ij}, i \in \{x, y, z\}$ and the deformation velocities, implies the existence of two viscosity coefficients: dynamic η and volumic η_V. Using cartesian coordinates, the relations between deformation tensions and velocities are:

$$\begin{cases} \tau_{xx} &= 2\eta\left(\frac{\partial v_x}{\partial x} - \frac{1}{3}\text{div } v\right) + \eta_v \text{div } v - p \\ \tau_{yy} &= 2\eta\left(\frac{\partial v_y}{\partial y} - \frac{1}{3}\text{div } v\right) + \eta_v \text{div } v - p \\ \tau_{zz} &= 2\eta\left(\frac{\partial v_z}{\partial z} - \frac{1}{3}\text{div } v\right) + \eta_v \text{div } v - p \end{cases} \quad (1)$$

$$\begin{cases} \tau_{xy} &= \tau_{yx} &= \eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \\ \tau_{yz} &= \tau_{zy} &= \eta\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \\ \tau_{zx} &= \tau_{xz} &= \eta\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right), \end{cases} \quad (2)$$

where v, with the cartesian components v_x, v_y, v_z is the velocity, p is the pressure and

$$\text{div } v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

is the divergence of the velocity.

These formulas represent the basis for the mathematical (deductive) study of viscous compressible and non-compressible fluids' movements [2].

The differential equations, in tensions, for newtonian fluid flow in non-stationary, isothermic conditions can be deduced from the equilibrium of: inertia $\rho dv/dt$, exterior ρf and surface $\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} + \frac{\partial \tau_z}{\partial z}$ forces for an element of volume. Under gravitational field, the force corresponding to an element of volume is ρg, where ρ is the density and g - the gravitation. Therefore, in cartesian coordinates we have:

$$\begin{aligned} \rho\frac{Dv_x}{Dt} &= \rho f_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\ \rho\frac{Dv_y}{Dt} &= \rho f_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\ \rho\frac{Dv_z}{Dt} &= \rho f_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}, \end{aligned}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}$ is the substantial derivative.

By substituting the formulas for tensions $\tau_{ij}, i, j \in \{x, y, z\}$ (1,2) in these last equations, we obtain the differential equations for newtonian fluid flow depending on the velocity components, known as the Navier-Stokes equations:

$$
\begin{aligned}
\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right) &= \eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)\\
&\quad+\left(\frac{\eta}{3}+\eta_v\right)\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_y}{\partial x\partial y}+\frac{\partial^2 v_z}{\partial x\partial z}\right)-\frac{\partial p}{\partial x}+\rho f_x\\
\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) &= \eta\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)\\
&\quad+\left(\frac{\eta}{3}+\eta_v\right)\left(\frac{\partial^2 v_x}{\partial y\partial x}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_z}{\partial y\partial z}\right)-\frac{\partial p}{\partial y}+\rho f_y\\
\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) &= \eta\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)\\
&\quad+\left(\frac{\eta}{3}+\eta_v\right)\left(\frac{\partial^2 v_x}{\partial z\partial x}+\frac{\partial^2 v_y}{\partial z\partial y}+\frac{\partial^2 v_z}{\partial z^2}\right)-\frac{\partial p}{\partial y}+\rho f_z
\end{aligned}
\tag{3}
$$

These equations describe the non-stationary isothermal flow of compressible fluids.

The vectorial form of the equations (3) is:

$$\rho\frac{Dv}{Dt}=\eta\Delta v+\left(\frac{\eta}{3}+\eta_v\right)\nabla(\nabla v)-\nabla p+\rho f,$$

where the ∇ and Δ operators are:

$$\nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k},\ \text{(Hamilton operator)}$$

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\ \text{(Laplace operator)}$$

The integration of Navier-Stokes equations (3) can be performed by analytical methods, which generate exact solution (but they often solve only limited particular cases), by numerical methods, which give approximate solutions and by experimental modelling methods, the latter being based on similarity and dimensional analysis. It is of utmost importance for the theoretical researchers to obtain exact, correct, symbolic or numeric results. Further on, we shall deal with this aspect.

3. Implementing the computations into Mathematica packages

Because of the complexity of the Navier-Stokes equations (3), they are solved in certain specific cases.

Thus, the Mathematica packages that were written are to compute, for certain types of movements, by means of the functions they contain, the physical quantities that characterize those movements, such as: velocity, friction tension, flow etc. These functions have a symbolic result, solving the problem from the theoretical point of view (the most important aspect for theoretical researchers). But if numerical arguments are wanted for some or all variables which appear in the computation of a certain quantity (function), they can be used and will generate an appropriate result.

Before presenting some examples, we mention that the numeric direct substitution of a differential or integration variable or of a solution for an equation (system of equations) is not, obviously, possible. To solve this problem, we defined auxiliary functions, which verified whether the argument matches a numeric pattern; if so, we applied a transformation rule upon the symbolic expression, substituting the variable by the numeric specified value. Another possible solution of the problem would have been defining the function as a block with local variables and similar effect.

Examples. A. *Stationary non-compressible fluid movements following concurrent lines. Specific case: movement between two plane walls*

In order to make the computations, we use cylindrical coordinates (x, r, ω), the components of the velocity being: $v_x = 0, v_r = v_r(r, \omega), v_\omega = 0$. From the continuity equation, we have $v_r = f(\omega)/r$, where f is a function depending on r. Navier-Stokes equations are (in cylindrical coordinates) [2]:

$$\begin{cases} 0 = -\frac{\partial p}{\partial x} \\ \rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial p}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \omega^2} - \frac{v_r}{r^2} \right] \\ 0 = -\frac{\partial p}{r \partial \omega} + \frac{2\mu}{r^2} \frac{\partial v_r}{\partial \omega} \end{cases} \tag{4}$$

We intend to find the formula for the pressure p, depending on f, r, η (dynamic viscosity coefficient) and an integration constant $C1$. The expression obtained after reducing equations (4) (integrated in respect to r, ω, respectively), will be named c; therefore we shall obtain the equation $c = 0$. The pressure p (which has the corresponding Mathematica function P) is deduced by integrating equation (6); subsequent to this operation we shall obtain function t, depending only on r. The velocity v_r has the corresponding Mathematica function V_r; all the others are auxiliary functions. Thus we obtain the following Mathematica functions:

```
Vr[r_,w_,f_]:=f[w]/r
A0[r_,w_,ro_,eta_,p_,f_]:=Integrate[ -ro Vr[r,w,f] D[Vr[r,w,f],r]-D[p[x,r,w],r]+
  eta(1/r D[r D[Vr[r,w,f],r],r]+D[Vr[r,w,f],{w,2}]/r^2-Vr[r,w,f]/r^2),r]
Axx0[r_,w_,ro_,eta_,p_,f_]:=If[ MatchQ[r,n_Integer] || MatchQ[r,n_Real],
    A0[rr,w,ro,eta,p,f] /. rr->r, A0[r,w,ro,eta,p,f]]
Ax0[r_,w_,ro_,eta_,p_,f_]:=If[ MatchQ[w,n_Integer] || MatchQ[w,n_Real],
    Axx0[r,ww,ro,eta,p,f] /. ww->w, Axx0[r,w,ro,eta,p,f]]
A1[r_,w_,eta_,p_,f_]:=Integrate[-D[p[x,r,w],w]/r + 2 eta/r^2 D[Vr[r,w,f],w],w]
Ax1[r_,w_,eta_,p_,f_]:=If[ MatchQ[w,n_Integer] || MatchQ[w,n_Real],
    A1[r,ww,eta,p,f] /. ww->w, A1[r,w,eta,p,f]]
c[w_,ro_,eta_,p_,f_]:=Simplify[Numerator[Simplify[
    -Ax0[r,w,ro,eta,p,f] + r Ax1[r,w,eta,p,f]]]/eta]
Axp0[r_,w_,eta_,f_,t_]:=Integrate[2 eta/r^2 f'[w],w]+t[r]
Axp[r_,w_,eta_,f_,t_]:=If[ MatchQ[w,n_Integer] || MatchQ[w,n_Real],
    Axp0[r,ww,eta,f,t] /. ww->w, Axp0[r,w,eta,f,t]]
Axt0[r_,w_,ro_,eta_,f_,t_]:=Simplify[(-ro Vr[r,w,f] D[Vr[r,w,f],r] -
```

```
  D[Axp[r,w,eta,f,t],r] + eta(1/r D[r D[Vr[r,w,f],r],r] +
  D[Vr[r,w,f],{w,2}]/r^2 - Vr[r,w,f]/r^2)) r^3/eta] (* ==0 ==>t *)
Axt1[r_,w_,ro_,eta_,f_,t_]:=If[ MatchQ[w,n_Integer] || MatchQ[w,n_Real],
     Axt0[r,ww,ro,eta,f,t] /. ww->w, Axt0[r,w,ro,eta,f,t]]
Axt[r_,w_,ro_,eta_,f_,t_]:=If[ MatchQ[r,n_Integer] || MatchQ[r,n_Real],
     Axt1[rr,w,ro,eta,f,t] /. rr->r, Axt1[r,w,ro,eta,f,t]]
ec2p[r_,w_,ro_,eta_,f_,t_]:=Simplify[ Numerator[ Simplify[
    Ax0[r,w,ro,eta,p,f] /. p[x,r,w] -> Axp[r,w,eta,f,t]]] / (-eta)]
taux[r_,w_,ro_,eta_,p_,f_,C1_]:=Simplify[((t[r] /. Part [ Solve[
   Integrate[(c[w,ro,eta,p,f]-Axt[r,w,ro,eta,f,t]) eta/r^3, r]
   ==Integrate[C eta/r^3,r] ,t[r]],1]) + C1) /. C->c[w,ro,eta,p,f]]
t[r_,w_,ro_,eta_,p_,f_,C1_]:=If[ MatchQ[r,n_Integer] || MatchQ[r,n_Real],
taux[rr,w,ro,eta,p,f,C1] /. rr->r, taux[r,w,ro,eta,p,f,C1] ]
P[r_,w_,ro_,eta_,f_,t_,C1_]:=Simplify[ Axp[r,w,eta,f,t] /. t[r]->t[r,w,ro,eta,p,f,C1]]
```

B. *Combined Couette-Poiseuille movement of newtonian media, with pressure gradient.*

Suppose the fluid flows between two parallel plates, one of them moves by velocity V, there is pressure gradient and a nucleus with $hb - ha$ thickness, which moves by constant velocity **vxc**. We use cartesian coordinates (x, y) and the velocity has only the component $v_x(y)$.

Velocity profile is given by Navier-Stokes equation [2]:

$$\frac{d^2 v_x}{dy^2} = \frac{1}{\eta}\frac{dp}{dx}$$

on the intervals $0 \le y \le ha, ha \le y \le hb$ and $hb \le y \le h$, with $v_x(0) = V$, $v_x(ha) = vxc$, $v_x(hb) = vxc$ and $v_x(h) = 0$. The friction tension τ_{xy} is given by:

$$\tau_{xy} = \mp\tau_0 + \eta\frac{\partial v_x}{\partial y},$$

where $\frac{dp}{dx}(ha - hb) = -2\tau_0$ and the flow has the usual formula [2]:

$$q_x = \int_0^x v_x dx.$$

We intend to calculate the velocity for the three intervals (**Vx1**, **Vx2**, **Vx3** in the Mathematica functions), the friction tension τ_{xy} for the intervals $0 \le y \le ha$ (*Tensxy1*) and $hb \le y \le h$ (**Tensxy3**) and the flow q_x (in Mathematica **Qx**). Thus we obtain the following Mathematica functions:

```
Aux1[y_,V_,ha_,eta_,p_,vxc_]:=Simplify[Collect[Part[vx[y] /.
    DSolve[{eta vx''[y]==p,vx[0]==V,vx[ha]==vxc},vx[y],y] ,1],{p,ha}]]
Aux3[y_,h_,hb_,eta_,p_,vxc_]:=Simplify[Collect[Part[vx[y] /.
    DSolve[{eta vx''[y]==p,vx[h]==0,vx[hb]==vxc},vx[y],y] ,1],{p,vxc}]]
Vx1[y_,V_,ha_,eta_,p_,vxc_]:=
    If[ MatchQ[p,_'[_]],
 If[ MatchQ[y,n_Integer] || MatchQ[y,n_Real],
     Aux1[yy,V,ha,eta,p,vxc] /. yy->y,
     Aux1[y,V,ha,eta,p,vxc]],
```

```
If[ (MatchQ[p,m_Integer] || MatchQ[p,m_Real] || MatchQ[p,n_Symbol])
    && (MatchQ[y,n_Integer] || MatchQ[y,n_Real]),
Aux1[yy,V,ha,eta,p,vxc] /. yy->y,
Aux1[y,V,ha,eta,p,vxc]]]
Vx2[vxc_]:=vxc
Vx3[y_,h_,hb_,eta_,p_,vxc_]:=
    If[ MatchQ[p,_'[_]],
 If[ MatchQ[y,n_Integer] || MatchQ[y,n_Real],
     Aux3[yy,h,hb,eta,p,vxc] /. yy->y,
     Aux3[y,h,hb,eta,p,vxc]],
 If[ (MatchQ[p,m_Integer] || MatchQ[p,m_Real] || MatchQ[p,n_Symbol])
    && (MatchQ[y,n_Integer] || MatchQ[y,n_Real]),
 Aux3[yy,h,hb,eta,p,vxc] /. yy->y,  Aux3[y,h,hb,eta,p,vxc]]]
Tensxy1[y_,V_,ha_,eta_,p_,vxc_]:=Simplify[Collect[
    If[ MatchQ[y,n_Integer] || MatchQ[y,n_Real],
Simplify[-eta D[Vx1[yy,V,ha,eta,p,vxc],yy]] /. yy->y,
Simplify[-eta D[Vx1[y,V,ha,eta,p,vxc],y]]],{eta,p}]]
Tensxy3[y_,h_,hb_,eta_,p_,vxc_]:=Simplify[Collect[
    If[ MatchQ[y,n_Integer] || MatchQ[y,n_Real],
Simplify[-eta D[Vx3[yy,h,hb,eta,p,vxc],yy]] /. yy->y,
Simplify[-eta D[Vx3[y,h,hb,eta,p,vxc],y]]],{eta,p}]]
Qx[V_,h_,ha_,hb_,eta_,p_,vxc_]:=Simplify[Apart[
    Integrate[Vx1[y,V,ha,eta,p,vxc],{y,0,ha}]+Integrate[vxc,{y,ha,hb}]+
    Integrate[Vx3[y,h,hb,eta,p,vxc],{y,hb,h}]]]
```

Subsequent to these calculations, we should consider the system of three equations obtained from the Navier-Stokes equation for the three specified intervals as a system with the unknown variables vxc, ha, hb depending on dp/dx and (0 and calculate their expressions.

An extension of the system of packages described above could be made by introducing input data as sets of numeric values characterizing the evolution of a quantity (velocity, for example), the expected result being the type of movement. Moreover, for an intuitive interpretation of the results, we can include in the packages functions which create a graphic representation of the expressions obtained for the computed quantities; this would not be much trouble considering Mathematica's facilities for drawing function plots.

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