STUDIA UNIV. "BABEŞ–BOLYAI", INFORMATICA, Volume I, Number 2, October 1996

THE NORMAL FORM OF A PERTURBED KEPLERIAN HAMILTONIAN

A. BLAGA

Abstract. In normalizing a perturbed Kepler Hamiltonian the Kepler Hamiltonian vector field is not complete, because solutions with angular momentum 0 reach the origin in *finite* time. We use first the *Moser regularization* and then obtain a *perturbed geodesic Hamiltonian*. In practice, computing the Poisson brackets will lead to discuss the constrained Hamiltonian systems and this paper deals with a genuine algorithmic method for computing a constrained normal form of a perturbed Keplerian Hamiltonian, involving the powerfull tool of Gröbner Bases Theory.

1. Introduction

Let f be a regularized perturbed Keplerian Hamiltonian. This paper describes an algorithm which brings f into normal form up to a certain order. A perturbed Keplerian Hamiltonian

$$f = H_0 + \varepsilon H_1 + \frac{\varepsilon^2}{2!} H_2 + \dots$$

is in normal form to order n iff $\{H_0, H_i\} = 0,\ i \in \{1, 2, \dots, n\}$, where H_0 is Kepler Hamiltonian which describes the motion of two bodies in $\mathbf{R}^3$ under the influence of the gravity.

For a general idea about regularization process see the appendix.

In the next section we will give a brief presentation of what a perturbed Kepler Hamiltonian is. At the end of section 3 we include the definition of a normal form.

The normal form algorithm starts in section 4 with some facts we need about Lie series and ends up with an important lemma about the space of Lie series. The mechanism of normal form algorithm can be found in section 5 but

Received by the editors: December 22, 1996.
1991 *Mathematics Subject Classification.* 13P10, 58F36, 68Q40.
1991 *CR Categories and Descriptors.* I.1.2 [Algebraic Manipulation]: Algorithms – algebraic algorithms.

the implementation techniques appears in the next section with the constrained normal form algorithm and the powerful tool of Gröbner bases ([1], [2]).

In the appendix one can follow the detailed presentation of the `MapleV` constrained normal form algorithm.

1.1. **The environment.** To explain our theory we need some background about Hamiltonian systems and perturbation theory. At the end of this section we will give the meaning of the perturbed Kepler Hamiltonian.

First of all let's consider $\mathbf{R}^3 - \{0\} \subseteq \mathbf{R}^3$, the space of positions $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbf{R}^3$, without the origin. On $\mathbf{R}^3$ define the euclidean inner product $< \xi, \eta >= \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$ and its induced norm $|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$.

On $T\mathbf{R}^3$ with coordinates (ξ, η) we have the standard symplectic form $\omega = \sum_{i=1}^{3} d\xi_i \wedge d\eta_i$.

Definition 1.1. A Hamiltonian *on* $(T\mathbf{R}^3, \omega)$ *is a* smooth *function*

$$H : T\mathbf{R}^3 \to \mathbf{R}$$

where the dynamics *of the system* $(T\mathbf{R}^3, \omega, H)$ *are the solutions of the differential equation*

$$\begin{cases} \dot{\xi} & = \dfrac{\partial H}{\partial \eta} \\ \dot{\eta} & = -\dfrac{\partial H}{\partial \xi}, \end{cases} \tag{1}$$

which are called Hamilton's equations.

Remark 1.2. *The solutions of the Hamilton's equations are the* integral curves *of the Hamiltonian* vectorfield X_H *on* $T\mathbf{R}^3$.

Definition 1.3. *The* Kepler Hamiltonian *describes the motion of two bodies in* $\mathbf{R}^3 - \{0\}$ *under the influence of gravity and is given by:*

$$H_0 : T_0\mathbf{R}^3 \to \mathbf{R}, \quad H_0(\xi, \eta) = \frac{1}{2}|\eta|^2 - \frac{\mu}{|\xi|}. \tag{2}$$

Here, $T_0\mathbf{R}^3 = (\mathbf{R}^3 - \{0\}) \times \mathbf{R}^3 \subseteq T\mathbf{R}^3$.

Definition 1.4. A perturbation H, of a Kepler Hamiltonian *of* H_0 *is a formal power series*

$$H = H_0 + \varepsilon H_1 + \frac{\varepsilon^2}{2!} H_2 + \ldots, \tag{3}$$

where H_i, $i > 0$ *are smooth functions on* $T_0\mathbf{R}^3$.

Before defining the normal form of a perturbed Keplerian Hamiltonian we will give a short description of what a Poisson algebra is.

2. Poisson algebra

Definition 2.1. *Let $(\mathcal{M}, \omega)$ be a smooth symplectic manifold. Let $\mathcal{F}(M)$ be the space of* smooth formal power series *in ε with coefficients in $C^\infty(\mathcal{M})$, the space of smooth functions on $\mathcal{M}$.*

Remark 2.2. *For $f, g \in C^\infty(\mathcal{M})$ define $f \cdot g \in C^\infty(\mathcal{M})$ by $(f \cdot g)(m) = f(m)g(m)$. Then $(\mathcal{F}(M), \cdot)$ is a commutative algebra.*

For $f, g \in C^\infty(\mathcal{M})$ define a *Poisson bracket* $\{\cdot,\cdot\}$ by

$$\{f, g\}(m) = \omega(m)(X_f(m), X_g(m)),$$

where X_f, X_g are Hamiltonian vector fields corresponding to f and g.

If $f = \sum_{i=0}^{\infty} f_i \varepsilon^i$, $g = \sum_{i=0}^{\infty} g_i \varepsilon^i$ in $\mathcal{F}(M)$ then define a Poisson bracket on $\mathcal{F}(M)$ by

$$\{f, g\} = \sum_{k=0}^{\infty} \Big(\sum_{i=0}^{k} f_i g_{k-i}\Big)\varepsilon^k.$$

Since $(C^\infty(\mathcal{M}), \cdot, \{\cdot,\cdot\})$ is a *Lie algebra*, $(\mathcal{F}(M), \cdot, \{\cdot,\cdot\})$ is also. Moreover, since

$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$$

for $f, g, h, \in C^\infty(\mathcal{M})$ it also holds for $f, g, h \in \mathcal{F}(M)$. Therefore $(\mathcal{F}(M), \cdot, \{\cdot,\cdot\})$ is a Poisson algebra.

Definition 2.3. *We define the* adjoint map*:*

$$ad_f : \mathcal{F}(M) \to \mathcal{F}(M),$$

by $ad_f g = \{f, g\}$, $f, g \in \mathcal{F}(M)$

Remark 2.4. *ad acts as a derivation on $(\mathcal{F}(M), \cdot)$.*

Definition 2.5. *Let H_0 be the Kepler Hamiltonian (2). The perturbed Kepler Hamiltonian $H = H_0 + \varepsilon H_1 + \frac{\varepsilon^2}{2!} H_2 + \dots$. H is in* normal form *iff $\{H_0, H_i\} = 0$ for all $i > 0$.*

In the finite case, meaning that $\{H_0, H_i\} = 0$ for $0 < i \leq n$, we say that H is in *normal form to order n*.

The main goal of this paper is to compute the normal form of a given degree n for a perturbed Kepler Hamiltonian, using the very nice idea from [5]. Unfortunately this idea has never been implemented in a computer algebra system. We start with the background theory of the algorithm.

3. Lie series and the splitting lemma

Definition 3.1. *If* $f \in \mathcal{F}(M)$ *then*

$$\varphi_\varepsilon^f = exp(\varepsilon ad_f) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} ad_f^n$$

is called Lie series.

Proposition 3.2. 1. *The formal flow of* X_f, $f \in \mathcal{F}(M)$ *is given by* $\varepsilon \mapsto \varphi_\varepsilon^f$.

2. φ_ε^f *is an automorphism of the formal Poisson algebra* $(\mathcal{F}(M), \cdot, \{\cdot, \cdot\})$:

$$\begin{aligned} \varphi_\varepsilon^f(gh) &= \varphi_\varepsilon^f(g) \cdot \varphi_\varepsilon^f(h), \\ \varphi_\varepsilon^f\{g, h\} &= \{\varphi_\varepsilon^f(g), \varphi_\varepsilon^f(h)\}. \end{aligned}$$

3. $\varepsilon \mapsto \varphi_\varepsilon^f$ *is one parameter group of automorphisms of formal Poisson algebra.*

Definition 3.3. X_f, $f \in C^\infty(\mathcal{M})$, *has periodic flow if there is* $T > 0$ *on* $\mathcal{M}$ *such that for every* $m \in \mathcal{M}$ *and* $g \in C^\infty(\mathcal{M})$

$$\varphi_{T(m)}^f g = g.$$

Remark 3.4. T *is not necessarily the minimal period of an integral curve of* X_f.

In the sequel we will give an important lemma, which is the basic tool in computing the normal form.

Lemma 3.5. *1. If* $H, F \in \mathcal{F}(M)$ *and* φ_ε^f *is the formal flow of* X_f, $f \in \mathcal{F}(M)$ *then*

$$(\varphi_\varepsilon^F)^* H = exp(\varepsilon ad_F) H.$$

2. **Splitting lemma.** *If* X_{H_0} *has periodic flow on* $C^\infty(\mathcal{M})$ *then*

$$C^\infty(\mathcal{M}) = \mathbf{ker}\ ad_{H_0} + \mathbf{im}\ ad_{H_0}. \tag{4}$$

Proof. The proof of the splitting lemma is based on solving the equation

$$L_{H_0} F = G, \quad F, G \in C^\infty(\mathcal{M}).$$

Given $F \in C^\infty(\mathcal{M})$ one can decompose it as $F = \overline{F} + (F - \overline{F})$, where $\overline{F}$ is the average:

$$\overline{F} = \frac{1}{T} \int_0^T ({\varphi_t}^{H_0})^* F \, dt. \tag{5}$$

Moreover, the equation $ad_{H_0} G = F - \overline{F}$ is solved by

$$G = \frac{1}{T} \int_0^T ({\varphi_t}^{H_0})^* (F - \overline{F}) \, dt. \tag{6}$$

The next section will deal with finding the normal form of a perturbed Kepler Hamiltonian. Section 6 gives the algorithms and several ideas in implementing the normal form.

4. Computing normal form

Before going into details we note that normal form theory finds a sequence of transformations which brings the perturbed Hamiltonian to normal form of a certain order.

A given $H = H_0 + \varepsilon H_1 + \frac{\varepsilon^2}{2!} H_2 + \dots$, $H \in \mathcal{F}(M)$ will be brought into a normal form up to some order as follows.

For $G_1 \in C^\infty(\mathcal{M})$ we have the Lie series, $\varphi_\varepsilon{}^{G_1} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} ad^n_{G_1}$. Then we change cordinates using $\varphi_\varepsilon^{G_1}$. The transformed Hamiltonian is:

$$\begin{aligned} F_\varepsilon^{(1)} &:= (\varphi_\varepsilon^{G_1})^* H = \\ &= H_0 + \varepsilon(H_1 + ad_{G_1} H_0) + \frac{\varepsilon^2}{2!}(H_2 + 2ad_{G_1} H_1 + ad^2_{G_1} H_0) + \mathcal{O}(\varepsilon^3). \end{aligned}$$

In the above equation we have to find $G_1 \in C^\infty(\mathcal{M})$.

By the splitting lemma H_1 can be decomposed as $H_1 = \overline{H_1} + H_1'$, where $\overline{H_1} \in \mathbf{ker}\ ad_{H_0}$ and $H_1' \in \mathbf{im}\ ad_{H_0}$. Thus

$$F_\varepsilon^{(1)} = H_0 + \varepsilon(\overline{H_1} + H_1' + ad_{G_1} H_0) + \frac{\varepsilon^2}{2!}(H_2 + 2ad_{G_1} H_1 + ad^2_{G_1} H_0) + \mathcal{O}(\varepsilon^3),$$

If we determine G_1 so that $H_1' = ad_{H_0} G_1$ then $F_\varepsilon^{(1)}$ is in normal form to the first order, since $\{H_0, \overline{H_1}\} = 0$. We now use (6) to determine G_1.

In order to compute the second order normal form it is important to preserve the already computed first order term from $F_\varepsilon^{(1)}$. Thus we have to use another change of coordinates, $\varphi_\varepsilon^{\varepsilon G_2}$. We obtain:

$$\begin{aligned} F_\varepsilon^{(2)} &:= (\varphi_\varepsilon^{\varepsilon G_2})^* F_\varepsilon^{(1)} = \\ &= H_0 + \varepsilon \overline{H_1} + \frac{\varepsilon^2}{2!}(H_2 + 2ad_{G_1} H_1 + ad^2_{G_1} H_0 + 2ad_{G_2} H_0) + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Here $G_2 \in C^\infty(\mathcal{M})$ should be determined by solving:

$$ad_{H_0} G = \tilde{H}_2{}',$$

where $\tilde{H}_2 = H_2 + 2ad_{G_1} H_1 + ad^2_{G_1} H_0$ and $\tilde{H}_2 = \overline{\tilde{H}_2} + \tilde{H}_2{}'$, with $\overline{\tilde{H}_2} \in \mathbf{ker}\ ad_{H_0}$ and $\tilde{H}_2{}' \in \mathbf{im}\ ad_{H_0}$.

In general having computed the $(n-1)^{th}$ order normal form, one could find the n^{th} order normal form by

$$F_\varepsilon^{(n)} := (\varphi_\varepsilon^{\varepsilon^{n-1} G_n})^* F_\varepsilon^{(n-1)},$$

where $G_1, G_2, \ldots, G_n - 1$ are known and $G_n(G_1, G_2, \ldots, G_n - 1)$ has to be solved.

One might use the nonrecursive formula:

$$F_\varepsilon^{(n)} := (\varphi_\varepsilon^{\sum_{i=1}^{n} \varepsilon^{i+1} G_i})^* H,$$

where $G_i, \quad i = 1, 2, \ldots, n$ is a *triangular* system of equations.

5. The constrained normal form algorithm

We introduce some notations.

Let

$$\begin{aligned} H_0(q,p) &:= \sqrt{|q|^2|p|^2 - <q,p>}, \quad (q,p) \in T\mathbf{R}^4 \text{ and} \\ T^+S^3 &= \{(q,p) \in T\mathbf{R}^4 \mid |q|^2 = 1, <q,p>= 0, p \neq 0\}, \end{aligned} \tag{7}$$

$$F_1 = |q|^2 - 1 = 0, \quad F_2 =< q,p >= 0. \tag{8}$$

Define the field $K := \mathbf{R}(|p|, k)$ of rational functions in $|p|$ and k, with real coefficients and its extension $K \subseteq L := \mathbf{R}(|p|, k, H_0)$.

If $(q,p) \in T\mathbf{R}^4$ we define

$$\mathcal{F} := \{f = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} f_n \mid f_n \in L[q,p]\}$$

and

$$\mathcal{G} := \{f \in \mathcal{F} \mid f_0 = H_0\}$$

Note that

$$\mathcal{F}|_{T^+S^3} = \{f = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} f_n \mid f_n \in K[q,p]\}$$

and

$$\mathcal{G}|_{T^+S^3} = \{f \in \mathcal{F}|_{T^+S^3} \mid f_0 = |p|\}.$$

Remark. For $f \in \mathcal{G}$, $f|_{T^+S^3} \in \mathcal{G}|_{T^+S^3}$ is a *perturbed geodesic Hamiltonian* ([5]).

The constrained normal form algorithm takes as input $f \in \mathcal{G}$ and the order of normalization and outputs $\tilde{f} \in \mathcal{G}$, the normal form of f with the required order.

From section 4 the first order normal form can be easily computed by averaging $f_1 \in K[q,p]$. For H_0 given by (7) we restrict the flow $\varphi_t^{H_0}$ to T^+S^3. It is given by:

$$\varphi_t^{H_0}|_{T^+S^3}(q,p) = \begin{pmatrix} q\cos 2t + \frac{p}{|p|}\sin 2t \\ -|p|q\sin 2t + p\cos 2t \end{pmatrix} \tag{9}$$

This reduces the amount of computation needed in averaging. Thus we calculate

$$\overline{f_1} = \frac{1}{\pi}\int_0^\pi f_1\left(q\cos 2t + \frac{p}{|p|}\sin 2t, -|p|q\sin 2t + p\cos 2t\right) dt. \tag{10}$$

After few steps of normalization the integrand in (8) will have so many terms that it will not be manageable. For this we introduce the equivalence relation $\approx$. For $f, g \in \mathcal{G}$, we say that $f \approx g$ iff $(f-g)|_{T^+S^3} = 0$. In other words $f \approx g$ iff f and g have the same coset representative N in $L[q,p]/I$, where I is the ideal generated by T^+S^3. Later we will see that N is called normal form of f or g. Practicaly speaking, to find a coset representative we have to introduce *Gröbner bases* for our problem:

Definition 5.1. *Let be $I \neq \{0\}$ an ideal in $L[q,p]$ and $G = \{g_1, g_2, \ldots, g_n\} \subseteq I$ a set of nonzero polynomials in $L[q,p]$. We define a* term ordering *on $L[q,p]$ by $q_1 > q_2 > q_3 > q_4 > p_1 > p_2 > p_3 > p_4$. G is a Gröbner basis for I iff for all $f \in I$, $f \neq 0$ there exists an $i \in \{1, 2, \ldots, n\}$ such that the leading monomial of g_i divides the leading monomial for f.*

Proposition 5.2. *If $G = \{g_1, g_2, \ldots, g_n\}$ is a Gröbner basis for I then I is generated by $g_1, g_2, \ldots, g_n$, that is*

$$I = < g_1, g_2, \ldots, g_n > .$$

If $f \in \mathcal{G}$ and $\mathcal{I} = < F_1, F_2 >$, then $r \in \mathcal{G}$, obtained by a division algorithm on multivariate polynomials with respect to the unique *reduced* Gröbner basis of I, it is also unique ([1]). $r \in \mathcal{G}$ is called the *normal form* of f and it is denoted by $N(f)$.

Proposition 5.3. *If $f, g \in L[q,p]$ then*

$$f \equiv g \pmod{I} \quad \text{iff} \quad N(f) = N(g).$$

Thus our definition for equivalence relation, $\approx$, can be restated as:

$$f \approx g \quad \text{iff} N(f) = N(g),$$

where the normal form is taken with respect to the Gröbner basis of the ideal $\mathcal{I}$ generated by T^+S^3.

Because T^+S^3 is an invariant manifold of X_{H_0} we see that $f \approx g$ implies $\overline{f} \approx \overline{g}$. This means that in order to compute the average of f we might compute $\overline{f}|_{T^+S^3}$ by using the Gröbner basis for the constraints. Thus, in our case we calculate $\overline{f_1} \in \mathcal{G}|_{T^+S^3}$ by just normalizing $\overline{f_1} \in \mathcal{G}$ with respect to our Gröbner basis.

The next step is preparing for the second order normal form. First we solve the equation $\overline{f_1} = ad_{H_0} g_1$ as it was pointed in section 4. This can be done by:

$$g_1 = \frac{1}{\pi} \int_0^{\pi} t\,(f_1 - \overline{f_1}) \left(q \cos 2t + \frac{p}{|p|} \sin 2t, -|p| q \sin 2t + p \cos 2t \right) dt.$$

In this case g_1 must *not* be normalized with respect to the Gröbner basis. In general, X_{g_1} does not leave T^+S^3 invariant. Thus g_1 has to be modified to g_1^* (see appendix):

$$g_1^* := g_1 - \frac{1}{2}\{g_1, < q, p >\}(|q|^2 - 1) + \frac{1}{2}\{g_1, |q|^2 - 1\} < q, p > .$$

Now $X_{g_1^*}$ leaves T^+S^3 invariant, meaning

$$\{g_1^*, < q, p >\}|_{T^+S^3} = 0$$

and

$$\{g_1^*, |q|^2 - 1\}|_{T^+S^3} = 0.$$

The last step in computing second order normal form is to use (6):

$$\tilde{f} = f_0 + \varepsilon\overline{f_1} + \frac{\varepsilon^2}{2!}\overline{(f_2 + 2ad_{g_1^*}f_1 + ad^2_{g_1^*}f_0)}.$$

Applying the same method one can compute higher order normal forms.

6. Appendix

6.1. **Regularization process.** In normalizing a perturbed Kepler Hamiltonian, some problems arise even at the very beginning: the Kepler Hamiltonian vector field is not complete. The reason for its incompleteness is that solutions with angular momentum 0 reach the origin in *finite* time.

To remove this incompletness we use *Moser regularization* ([3]), which changes the phase space from $T_0\mathbf{R}^3$ to $T(S^3 - (0,0,0,1))$, the tangent bundle of the sphere $S^3 \subseteq \mathbf{R}^4$ without the north pole.

First of all we introduce the time scale by $\frac{ds}{dt} = \frac{k}{|\xi|}$ and define

$$M(\xi, \eta) = \frac{|\xi|}{k}\left(H(\xi, \eta) + \frac{k^2}{2}\right) + \frac{\mu}{k},$$

where μ comes from (2). The Hamiltonian vector field of (2) is clearly:

$$\begin{cases} \dfrac{d\xi}{dt} &= \eta \\ \dfrac{d\eta}{dt} &= -\mu\dfrac{\xi}{|\xi|^3}. \end{cases} \tag{11}$$

We would like to have rescaled Hamiltonian equations for H, so we restrict (ξ, η) to lie in the level set $H^{-1}(-k^2/2)$ or equivalently $M^{-1}(\mu/k)$, $\mu/k > 0$.

Remark 6.1. *$H < 0 \Rightarrow H_0 < 0$, so we have bounded Keplerian orbits.*

Thus, after time rescaling, the equations satisfied by M on the level set $M^{-1}(\mu/k)$, are:

$$\begin{cases} \dfrac{d\xi}{ds} = \dfrac{|\xi|}{k}\dfrac{\partial H}{\partial \eta} \\ \dfrac{d\eta}{dt} = -\dfrac{|\xi|}{k}\dfrac{\partial H}{\partial \xi}. \end{cases} \tag{12}$$

As a last step of regularization, compose M with *Moser's map*:

$$\mathcal{M} : T(S^3 - (0,0,0,1)) \to T_0\mathbf{R}^3, \quad \mathcal{M}(q,p) = (\xi,\eta),$$

$$\xi_i = -\frac{1}{k}(p_i + q_i p_4 - p_i q_4),$$

$$\eta_i = \frac{kq_i}{1-q_4}, \quad i = 1,2,3.$$

The new Hamiltonian is:

$$G(q,p) = G_0(p) + \varepsilon G_1(q,p) + \frac{\varepsilon^2}{2!}G_2(q,p) + \cdots,$$

where $G_0(q,p) = |p|$. Thus a perturbed Kepler Hamiltonian is transformed into a *perturbed geodesic Hamiltonian* with its geodesic vector field X_{G_0} on

$$T^+S^3 = \{(q,p) \in T\mathbf{R}^4 |\ |q|^2 = 1, < q,p >= 0, p \neq 0\}.$$

$|q|^2 = 1$ is the condition for the body to move on the unit sphere. For it to have phase space TS^3 we need the extra condition $< q,p >= 0$.

Another problem that arises in practice is computing Poisson brackets on T^+S^3. This leads to the discussion of *constrained Hamiltonian systems*.

6.2. **Constraints.** The manifold $TS^3 \subseteq T\mathbf{R}^4$ defined by:

$$\begin{cases} F_1(q,p) = |q|^2 - 1 = 0 \\ F_2(q,p) = < q,p >= 0 \end{cases}$$

is called *constraint manifold*. Since the matrix

$$\begin{pmatrix} \{F_1,F_1\} & \{F_1,F_2\} \\ \{F_2,F_1\} & \{F_2,F_2\} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

on TS^3 is non-singular, TS^3 is a symplectic manifold with symplectic form $\omega = \omega|_{TS^3}$. Since $T^+S^3 \subseteq TS^3$ is an open set, it is also a symplectic manifold.

In the sequel we will give a method to compute the Poisson brackets $\{,\}_{TS^3}$ on $C^\infty(TS^3)$ by $\{f,g\}_{TS^3} = \omega(X_f, X_g)$. The idea is to construct smooth functions F^* and G^*, which are extensions of f and g on $T\mathbf{R}^4$ so that:

$$\{f,g\}_{TS^3} = \{F^*,G^*\}|_{TS^3}, \tag{13}$$

and

$$F^*|_{TS^3} = f, \quad G^*|_{TS^3} = g.$$

Let F be an arbitrary smooth extension of f to $\mathbf{R}^4$. This extension exists by *Whitney extension theorem* ([4]), because TS^3 is closed subset of $T\mathbf{R}^4$.

Because TS^3 is not an *invariant manifold* if X_f we can not use F to compute $\{,\}_{TS^3}$. We have to modify F. Let $F^* = F + \alpha_1 F_1 + \alpha_2 F_2$ and choose $\alpha_1, \alpha_2 \in C^\infty(\mathbf{R}^4)$ so that $\{F^*, F_1\}|_{TS^3} = 0$ and $\{F^*, F_2\}|_{TS^3} = 0$. This means that TS^3 is invariant manifold of X_{F^*}. So, for $F \in C^\infty(T\mathbf{R}^4)$

$$F^* = F - \frac{1}{2}F_1 + \frac{1}{2}F_2,$$

the Poisson bracket $\{,\}_{TS^3}$ on $C^\infty(TS^3)$ is given by

$$\{F|_{TS^3}, G|_{TS^3}\}_{TS^3} = \{F^*, G^*\}|_{TS^3}.$$

Now we look at $H_0 = \sqrt{|p|^2|q|^2 - <q,p>^2}$. H_0 is smooth on the symplectic submanifold $\tilde{\mathcal{M}} = T\mathbf{R}^4 - \{H_0 = 0\}$ of $(T\mathbf{R}^4, \omega)$. Clearly $H_0|_{T^+S^3} = |p|$ and T^+S^3 is an invariant manifold of X_{H_0}. Another fact is that the flow $\varphi_t^{H_0}$ on $\tilde{\mathcal{M}}$ is given

$$\varphi_t^{H_0}(q,p) = \begin{pmatrix} -\frac{<q,p>}{H_0}\sin 2t + \cos 2t & \frac{|q|^2}{H_0}\sin 2t \\ -\frac{|p|^2}{H_0}\sin 2t & \frac{<q,p>}{H_0}\sin 2t + \cos 2t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$

Clearly $\varphi_t^{H_0}$ is periodic with period π. See [6] for more details.

References

[1] W.W. Adams, P. Loustanau, *An Introduction to Gröbner Bases*, American Mathematical Society, 1994.

[2] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties and Algorithms: an introduction to computational algebraic geometry and commutative algebra*, Springer-Verlag, Berlin, 1992.

[3] R. Cushman, *Normal Form for Hamiltonian Vectorfields with Periodic Flow* in "*Diffferential Geometric Methods in Mathematical Physics*", pp. 125-144, ed. Sternberg, Reidel, Dordrecht, 1984.

[4] R. Cushman, J.A. Sanders, *The Constrained Normal Form algorithm*, Celestial Mechanics 45, pp. 181-187, 1989.

[5] R. Cushman, *A Survey of Normalization Techniques Applied to Perturbed Keplerian Systems*, Utrecht University, Department of Mathematics, preprint 598, 1990.

[6] R. Cushman, R. Bates, *Global Aspects of Classical Integrable Systems*, to appear.

[7] J,C. van der Meer, R. Cushman, *Constrained Normalization of Hamiltonian Systems and Perturbed Keplerian Motion*, Journal of Applied Mathematics and Physics (ZAMP), volume 37, pp. 402-424, 1986.

[8] J.C. van der Meer, R. Cushman, *Orbiting Dust under Radiation Pressure* in "*Diffferential Geometric Methods in Mathematical Physics*", pp. 403-414, ed. Doebner, World Scientific, Singapore, 1987.

[9] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics 107, Springer-Verlag, New York, 1986.

"Babeş-Bolyai" University, Faculty of Mathematics and Informatics, RO-3400 Cluj-Napoca, Romania

E-mail address: blaga@cs.ubbcluj.ro