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A PROOF METHOD FOR CLOSED NORMAL DEFAULT THEORIES

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Abstract. In his paper [4], Reiter defined a proof method for closed normal theories, based on the linear resolution for classical logic. A refinement of the Reiter method, using linear resolution with ordered clauses is proposed in this paper. This refinement is exemplified by some examples.

1. Introduction

Reiter introduced in [4] a very attractive formalization of the non-monotonic reasoning, adding to the first-order logic a new kind of inference rules called defaults. A default is an expression of the form $\frac{\alpha(x):\beta_1(x),\ldots,\beta_m(x)}{\gamma(x)}$, and is interpreted as follows : "if one believes $\alpha(x)$ and it is consistent to believe $\beta_1(x),\ldots,\beta_m(x)$, then one can also believe $\gamma(x)$ ".
The default logic presented in this paper is taken from [3] and [4].

Definition 1.1. *Let L be a first-order language. A* ***default theory*** *$\Delta=(D,W)$ consists of a set W of closed formulas in first-order logic and a set D of defaults $d=\frac{\alpha(x):\beta_1(x),\ldots,\beta_m(x)}{\gamma(x)}$, where*
-$\alpha(x),\beta_1(x),\ldots,\beta_m(x),\gamma(x)$ are well formed formulas in first-order logic;
-$\alpha(x)$ is called the ***prerequisite*** *of the default d;*
-$\beta_1(x),\ldots,\beta_m(x)$ are called ***justifications*** *of the default d;*
-$\gamma(x)$ is called the ***consequent*** *of the default d.*

The set W of formulas is the based knowledge of the theory, are treated as axioms, and we may reason about them using the inference rules of classical logic and the defaults, obtaining **beliefs**.
The defaults are inference rules which complete the incompletely specified world W. These rules model the non-monotonic reasoning, thus, we can infer conclusions,

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later on invalidated by new informations which denied the justifications of some defaults already applied.

A default theory $\Delta = (D, W)$ is **consistent** if W is a consistent set of formulas.
A **default is closed** if none of $\alpha(x), \beta_1(x), \ldots, \beta_m(x), \gamma(x)$ contains a free variable. If the **default** is not closed, it is **open**. A **default theory is closed** if all of its defaults are closed.
Every open default theory may be transformed into a closed theory. Therefore in all what follows we will only consider the closed default theories. A closed default has the form $\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma}$.
We denote
$PRE(D) = \bigcup\{\alpha|\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D\}$, and
$CONS(D) = \bigcup\{\gamma|\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D\}$.

Having a default theory we are interested to extend it (using the inference rules from classical logic and the defaults), in a consistent way, obtaining all the acceptable set of beliefs that one may hold about the incompletely specified world W.

Definition 1.2. *Let $\Delta = (D, W)$ be a closed default theory. For any set of closed formulas $S \subseteq L$, let $\Gamma(S)$ be the smallest set satisfying the following properties: (i) $W \subseteq \Gamma(S)$;*
(ii) $Th(\Gamma(S)) = \Gamma(S)$, where $Th(S) = \{P|S \vdash P\}$;
(iii) if $\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma} \in D$ and $\alpha \in \Gamma(S)$ and $\neg\beta_1, \ldots, \neg\beta_m \notin S$ then $\gamma \in \Gamma(S)$.
*A set of closed formulas $E \subseteq L$ is an **extension** for Δ if and only if $\Gamma(E) = E$, i.e. E is a fixed point of the operator Γ.*

The default theories can have zero, one or more extensions.

Definition 1.3. *The set of **generating defaults** of an extension E of a closed default theory is*
$GD(E, \Delta) = \{d \in D | d = \frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma}$ *and* $\alpha \in E$ *and* $\neg\beta_1, \ldots, \neg\beta_m \notin E\}$

Theorem 1.4. *Let E be an extension of the closed default theory $\Delta = (D, W)$. Then $E = Th(W \cup CONS(GD(E, \Delta)))$.*

2. Normal default theories

Between default theories, there is a class of theories which have always an extension. These theories have all their defaults of the form $\frac{\alpha:\beta}{\beta}$, and are called **normal default theories**. Their defaults are called **normal defaults** and they model the most common non-monotonic rules used in practice :"if α, then believe β until the contrary of β is known". The normal default theories have many interesting properties according to the following theorems.

Theorem 2.1. *Every closed normal default theory has an extension.*

Theorem 2.2. **(Semi-monotonicity)** *Let $\Delta = (D, W)$ and $\Delta' = (D', W)$ be two normal default theories, $D' \subseteq D$, and E' an extension of Δ'. Then the theory Δ has an extension E such that:*
(i) $E' \subseteq E$;
(ii) $GD(E', \Delta') \subseteq GD(E, \Delta)$.

(i) says that the normal default theories are monotonic with respect to defaults. An important practical consequence of (ii) is that closed normal default theories admit a proof procedure which is local with respect to the defaults, so that proofs may be constructed which ignore some of the defaults.

Theorem 2.3. *(Orthogonality of extensions) If E and F are two extensions of the same normal default theory, then $E \cup F$ is an inconsistent set of formulas.*

Theorem 2.4. *Suppose $\Delta = (D, W)$ is a closed normal default theory such that $W \cup CONS(D)$ is consistent. Then Δ has a unique extension.*

In classical logics and monotonic logics all the formulas derived are valid, they are called theorems. For non-monotonic logics, a derived formula (a **belief**) is not necessary valid, it is only consistent with all the formulas of the extension to which belongs.
A proof theory for the default theories means a method for answering the question "given β, can β be believed?". This question may be formalized in :"given a closed normal default theory Δ and a closed formula $\beta \in L$, determine whether Δ has an extension E such that $\beta \in E$."

Definition 2.5. *(Reiter [4]) Let $\Delta = (D, W)$ be a closed normal default theory and $\beta \in L$ a closed formula. A finite sequence $D_0, \ldots, D_k$ of finite subsets of D is a* ***default proof of*** *β with respect to Δ if and only if :*
(p1) $W \cup CONS(D_0) \vdash \beta$;
(p2) for $1 \leq i \leq k$
$W \cup CONS(D_i) \vdash PRE(D_{i-1})$;
(p3) $D_k = \emptyset$;
(p4) $W \cup \bigcup_{i=0}^{k} CONS(D_i)$ is consistent.

This definition does not provide a real proof procedure, because it gives no method for determining the D_i sets, nor does it specify a method to verify the consistency of a set of formulas. We may say that if the conditions *(p1)*, *(p2)*, *(p3)*, *(p4)* are satisfied, then β can be believed.

The completeness of this method results from the following two theorems.

Theorem 2.6. *Let $\Delta = (D, W)$ be a consistent closed normal theory, and $\beta \in L$ a closed formula. If β has a default proof $D_0, \ldots, D_k$ with respect to Δ, then Δ has an extension E such that $\beta \in E$.*

Theorem 2.7. *Suppose that E is an extension for a consistent closed normal default theory $\Delta = (D, W)$, and that $\beta \in E$. Then β has a default proof with respect to Δ.*

The condition *(p4)* is a test of consistency for a set of formula in first-order logic, but we know that this problem is not semi- decidable. Therefore we have the following theorem:

Theorem 2.8. *The extension membership problem for closed normal default theories is not semi-decidable.*

A proof method for normal default theories, inspired from the Reiter method is defined and studied in [5].

3. Linear resolution and OL-resolution

3.1. Linear resolution. Linear resolution, introduced by Loveland, provides a top down theorem prover. This refinement of resolution is complete and was used by Reiter ([4]) in his default proof method to obtain the D_i sets.

Definition 3.1. *A* ***linear resolution proof of*** *β from some set of clauses S has the form of figure1, where:*
(i) the ***top clause*** *R_0 is a clause of $\neg\beta$*
(ii) for $1 \leq i \leq n$, R_i (called a ***center clause****) is a resolvent of R_{i-1} and C_{i-1} (called a* ***side clause****)*
(iii) for $0 \leq i \leq n-1$, $C_i \in S$ or C_i is a clause of $\neg\beta$ or C_i is R_j for some $j < i$.
(iv) $R_n = \square$, the empty clause.

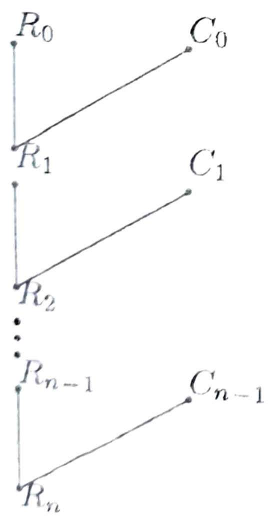

FIGURE 1. Linear resolution

3.2. **Ordered linear resolution.** We can strengthen the linear resolution by ordering the clauses and by using the information of the literals resolved upon, and we obtain the **ordered linear resolution**(OL-resolution). These two concepts increase the efficiency of linear resolution and do not destroy its completeness. An **ordered clause** means that its literals are increasing ordered according to their position in the clause, from left to right. In the process of linear deduction we can require that the literal resolved upon from the center clause be the last in the clause.
The informations of the literals resolved upon allow to know if a side clause is a clause of the initial set or a center clause previously generated. We can keep the literal resolved upon from the center clause in the resolvent and it becomes **framed literal**.

Example 3.2. *The resolvent of the ordered clauses* $C_1 = P \vee Q$ *and* $C_2 = P \vee \neg Q \vee R$ *is the clause* $C = P \vee [Q] \vee P \vee R$.
We can apply ***merging left*** *operation for identical unframed literals and the clause* C *becomes* $P \vee [Q] \vee R$.

In this way we can lead the process of deduction, eliminating the expansive search of the side clauses.
These concepts are formalised in the following definitions and theorems taken from [1] and [2].

Definition 3.3. *An ordered clause* R *is a* ***reducible ordered clause*** *if and only if the last literal of* R *is unifiable with the negation of a framed literal of* R.

Definition 3.4. *Let* R_1 *and* R_2 *be two ordered clauses with no variables in common and* L_1, L_2 *be two unframed literals in* R_1 *and* R_2 *respectively. Let* L_1 *and* $\neg L_2$ *have a most general unifier* δ. *Let* R^* *be the ordered clause obtained by concatenating the sequence* $R_1\delta$ *and* $R_2\delta$, *framing* $L_1\delta$, *removing* $L_2\delta$, *and merging left for any identical unframed literals in the remaining sequence. Let* R *be obtained from* R^* *by removing every framed literal not followed by any unframed literal in* R^*. R *is called an* ***ordered resolvent*** *of* R_1 *against* R_2. *The literals* L_1 *and* L_2 *are called the* ***literals resolved upon***.

Definition 3.5. *Let* R *be a reducible ordered clause. Let the last literal* L *be unifiable with the negation of some framed literal with a most general unifier* δ. *The* ***reduced ordered clause of R*** *is the ordered clause obtained from* $C\delta$ *by deleting* $L\delta$ *and every subsequent framed literal not followed by any unframed literal.*

Definition 3.6. *Given a set* S *of ordered clauses and an ordered clause* R_0 *in* S, *an* ***OL-deduction of*** R_n ***with top ordered clause*** R_0 *is a deduction which satisfies the conditions:*

(i) For i=0,1,...,n-1, R_{i+1} *is an ordered resolvent of* R_i *(called a* ***center ordered clause****) against* C_i *(called* ***side ordered clause****). The literal resolved upon*

in R_i is the last literal.

(ii) Each C_i is either an ordered clause in S or an instance of some R_j, $j < i$. C_i is an instance of some C_j, $j < i$, if and only if R_i is a reducible ordered clause. In this case, R_{i+1} is the reduced ordered clause of R_i.

(iii) No tautology is in the deduction.

Theorem 3.7. ***(Completeness of OL-deduction)*** *If R is an ordered clause in an unsatisfiable set S of ordered clauses, then there is an OL-deduction of the empty clause from S with top ordered clause R.*

Example 3.8. *Let $S = \{P \vee Q, P \vee \neg Q \vee R, P \vee \neg Q \vee \neg R, \neg P \vee R, \neg P \vee \neg R\}$ be a set of clauses. We use OL-deduction to proved that the set S is unsatisfiable. The figure2 represents the OL-deduction of $\Box$ from S with $P \vee Q$ as a top clause.*

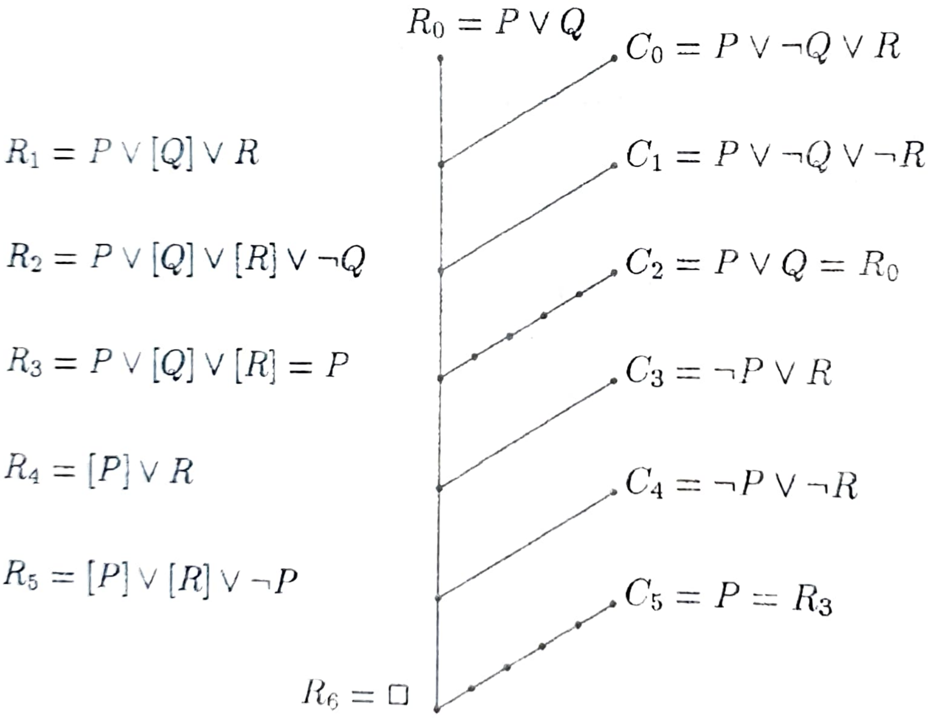

FIGURE 2. OL-deduction

In this example, R_2 is a reducible ordered clause, therefore C_2 is a center clause previously generated, namely R_0, and R_3 is the reduced ordered clause of R_2. Similar R_6 is the reduced ordered clause of R_5.
The steps 3 and 6 in the process of deduction can be eliminated, constructing directly the reduced ordered clauses R_3 and R_6. We can observe that the framed literals are useful only for the unframed literals which follow them.

This refinement of linear resolution is very easy to implement and is very efficient, because we need only the last resolvent (last center clause) in the process of deduction.

4. A default proof method for normal default theories using OL-resolution

A refinement of the Reiter proof method [4] for closed normal theories is proposed in this section. The OL-resolution seems to be appropriate to be used in the default proof method of Reiter for determine the sets D_i. The idea is to use OL-resolution to allow the goal β to help select a suitable subset D_0 of D, and so on. We must have an appropriate representation of a closed normal default theory for using the OL-resolution. Assume that W is a set of ordered clauses. For a normal default $d = \frac{\alpha:\gamma}{\gamma}$ suppose $C_1, ..., C_r$ are all the clauses of γ. A pair $(C_i, \{d\})$ is called an **ordered consequent clause** of the default d.

Let $\Delta = (D, W)$ be a closed normal default theory, where W is a set of ordered clauses. We define $CLAUSES(\Delta) = \{(C, \emptyset) | C \in W\} \cup$
$\cup \{(C, \{d\}) | d \in D$ and $(C, \{d\})$ is an ordered consequent clause of d$\}$.

A pair *(C,D)*, where C is an ordered clause and D is a set of defaults is called **indexed ordered clause**; C is said to be **indexed by *D***.

The **resolvent** of the two indexed ordered clauses (C_1, D_1) and (C_2, D_2) is the indexed ordered clause $(R, D_1 \cup D_2)$, where R is the ordered resolvent of C_1 against C_2.

An OL-resolution of β from some set S of indexed ordered clauses has the properties:

- the top clause R_0 is an ordered clause of $\neg\beta$;
- for $1 \leq i \leq n$, R_{i-1} and C_{i-1} are indexed ordered clauses and R_i is their resolvent;
- for $0 \leq i \leq n-1$, $C_i \in S$ or C_i is a an ordered clause of $\neg\beta$ or C_i is R_j for some $j < i$;
- $R_n = (\square, D)$ for some set of defaults.

We say that such OL-resolution proof of β returns D.

Definition 4.1. *A top down default proof of β with respect to a closed normal default theory $\Delta = (D, W)$ is a sequence of OL- resolution proofs $L_0, ..., L_k$ such that:*

(i) L_0 is an OL-resolution proof of β from $CLAUSES(\Delta)$;
(ii) for $0 \leq i \leq k$, L_i returns D_i;
(iii) for $1 \leq i \leq k$, L_i is an OL-resolution proof of $PRE(D_{i-1})$ from $CLAUSES(\Delta)$;
(iv) $D_k = 0$;
(v) $W \cup \bigcup_{i=0}^{k} CONS(D_i)$ is consistent.

Theorem 4.2. *((Completeness of top down default proofs) Let $\Delta = (D, W)$ be a closed normal default theory, and β a closed formula. Then the theory has an extension E such that $\beta \in E$ if and only if β has a top down default proof with respect to the theory.*

The demonstration of this theorem results from the completeness of default proofs (Theorem 2.6 and Theorem 2.7) and the completeness of OL-resolution (Theorem 3.7). **Remarks:**
1. The defaults of $D_0, D_1, \ldots, D_{k-1}$ belong to the generating defaults set of the extension E.
2. If there are more extensions which contain the same formula β, then exists a top down default proof for β corresponding to each extension.
3. If none of the defaults of the theory has prerequisite, then the top down default proof consists in L_0 only.

Example 4.3. *Let $\Delta = (D, W)$ be a normal default theory, where $W = \{\neg P \vee R, P \vee \neg Q \vee \neg R\}$, and $D = \{d_1 = \frac{:P\vee\neg Q\vee R}{P\vee\neg Q\vee R}, d_2 = \frac{:\neg P\vee\neg R}{\neg P\vee\neg R}\}$.*
A top down default proof of $\beta = \neg P \wedge \neg Q$ is L_0 which returns $D_0 = \{d_1, d_2\}$ according to figure3.

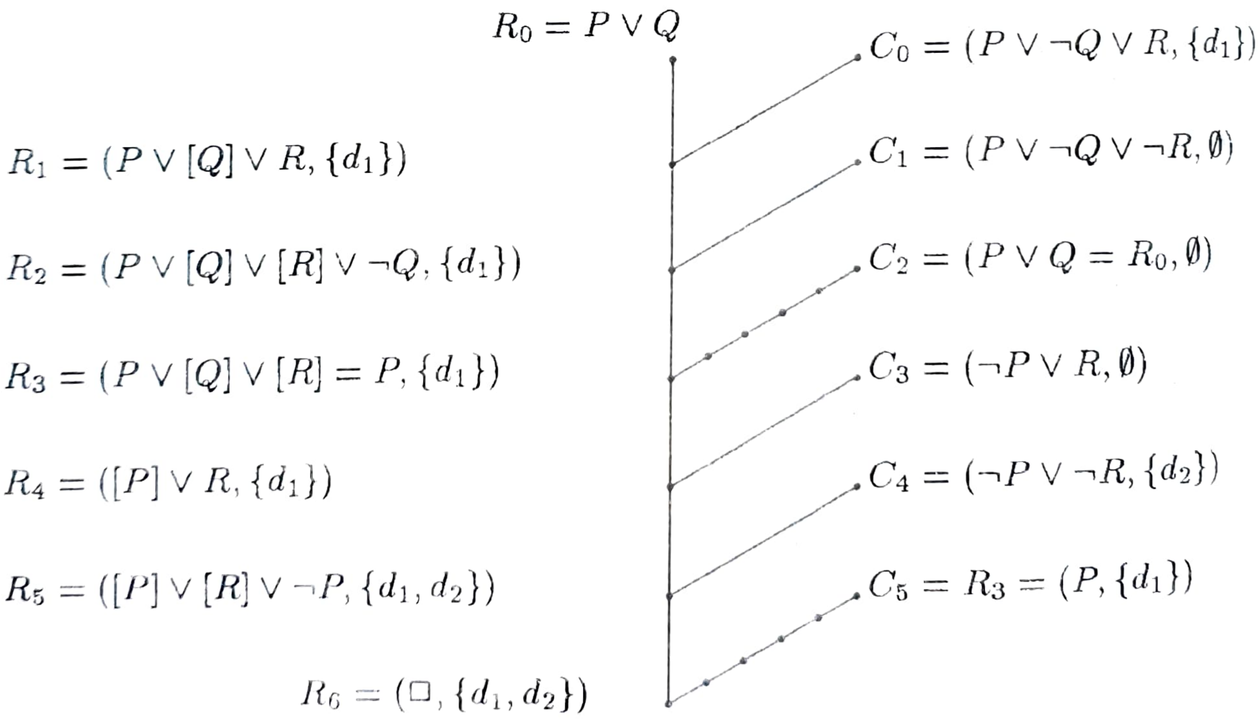

FIGURE 3. A top down default proof of $\neg P \wedge \neg Q$

In this example R_0 is $\neg\beta$, $CLAUSES(\Delta) = \{(\neg P \vee R, \emptyset), (P \vee \neg Q \vee \neg R, \emptyset), (P \vee \neg Q \vee R, \{d_1\}), (\neg P \vee \neg R, \{d_2\})\}$. The set $W \cup CONS(\{d_1, d_2\})$ is consistent. The theory of this example has an unique extension $E = Th(W \cup CONS(\{d_1, d_2\}))$, and $\beta \in E$.

Example 4.4. *Let $\Delta = (D, W)$ be a closed normal default theory, where $W = \{C \to D, A \land B \to E, E \lor D, D \to F\}$, and $D = \{d_1 = \frac{E \lor F : A \land F}{A \land F}, d_2 = \frac{A : B}{B}, d_3 = \frac{A \land E : C}{C}, d_4 = \frac{: \neg E}{\neg E}\}$.*
This theory has two extension $E_1 = Th(W \cup \{A \land F, B, C\})$ and $E_2 = \{Th(W \cup \{A \land F, \neg E\})$. We want to demonstrate that formula $\beta = D \land A$ belongs to both extensions. We will construct the top down default proof of β corresponding to extension E_2 (figure 4).

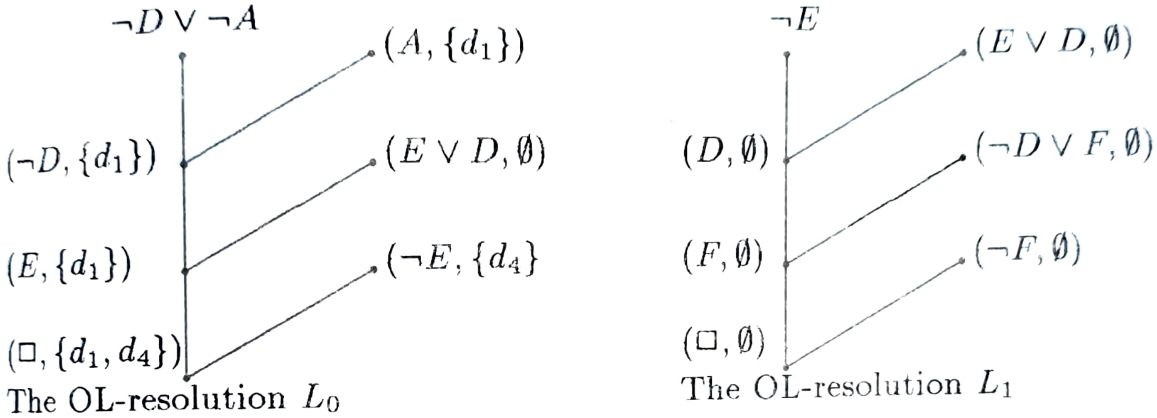

FIGURE 4

$CLAUSES(\Delta) = \{(A, \{d_1\}), (F, \{d_1\}), (B, \{d_2\}), (C, \{d_3\}), (\neg E, \{d_4\}), (\neg C \lor D, \emptyset), (\neg A \lor \neg B \lor E, \emptyset), (E \lor D, \emptyset), (\neg D \lor F, \emptyset)\}$

The top down default proof of $A \land D$ as an element of E_2 is the sequence L_0, L_1, where:

L_0 returns $D_0 = \{d_1, d_4\}$, $PRE(D_0) = \{\neg E, \neg F\}$
L_1 returns $D_1 = \emptyset$.
The set $W \cup CONS(\{d_1, d_4\})$ is consistent.
The top down default proof of $A \land D$ as an element of E_1 is the sequence L_0, L_1, L_2, L_3 where:

L_0 returns $D_0 = \{d_1, d_3\}$
L_1 returns $D_1 = \{d_2, d_1\}$
L_2 returns $D_2 = \{d_1\}$
L_3 returns $D_3 = \emptyset$
and can be constructed in a similar way. The set $W \cup CONS(\{d_1, d_2, d_3\})$ is consistent.

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